Analisi matematica IV - TLC 2/2/09

1) Assegnato $y \in \mathbb{R}^3 : \|y\| < 1$, individuare la funzione $f : x \in \mathbb{R}^3 - \{y\} \to f(x) \in]0, +\infty[$ tale che

$$\| y + f(x)(x-y) \| = 1 \quad \forall x \in \mathbb{R}^3 - \{y\}.$$

2) Si ricostruisca la funzione razionale $g(z)$ sapendo che

a) $g(0) = 1$, b) g ha solo due poli, uno semplice in $z = -1$, con residuo 1, ed uno doppio in $z = 1$, con residuo 1, c) $\lim_{z \to \infty} g(z) = 2$

3) Usando la trasformata di Laplace, risolvere il seguente problema:

$$\begin{cases} y'' - y' = 1 - t \\ y(0) = 0 \\ y(1) = \frac{1}{2} \end{cases}$$

4) Provare che

$$\mathcal{F}\left[\frac{1}{x^4+1}\right](\omega) = \frac{\pi}{\sqrt{2}} e^{-\frac{\omega}{\sqrt{2}}} \left(\cos\left(\frac{\omega}{\sqrt{2}}\right) + \operatorname{sen} \frac{\omega}{\sqrt{2}} \right) \quad \forall \omega > 0$$

(Usare il lemma di Jordan).

5) Assegnato $b \in]0, 1[$, ~~calcolare~~ dimostrare che

$$\int_{-\pi}^{\pi} \frac{\operatorname{sen} \vartheta}{1 + b \cos \vartheta} \, d\vartheta = 0,$$

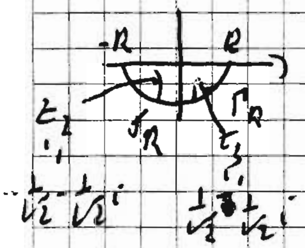
usando tecniche di analisi complessa.

1) $f(x) \overset{?}{\exists}\, t \quad \|y + t(x-y)\|^2 = 1 \iff \|y\|^2 + t^2\|x-y\|^2 + 2t\,y(x-y) = 1 \iff$
$t^2\|x-y\|^2 + 2y(x-y)t + \|y\|^2 - 1 = 0 \iff t = \dfrac{-y(x-y) + \sqrt{(y(x-y))^2 - (\|y\|^2-1)\|x-y\|^2}}{\|x-y\|^2}$

2) $y(t) = \frac{1}{z+1} + \frac{1}{z-1} + \frac{\alpha}{(z-1)^2} + 2, \quad y(0) = 1 \Rightarrow 1 - 1 + \alpha + 2 = 1 \iff \alpha = -1$

3) $s^2\mathcal{L}[y] - \alpha - s\mathcal{L}[y] = \frac{1}{s} - \frac{1}{s^2} \iff \mathcal{L}[y] = \left(\frac{1}{s} - \frac{1}{s^2} + \alpha\right)\frac{1}{s^2 - s} = \frac{s - 1 + s^2\alpha}{s^2}\,\frac{1}{s^2-s} =$
$= \frac{1}{s^3} + \frac{\alpha}{s(s-1)} = \frac{1}{s^3} - \frac{\alpha}{s} + \frac{\alpha}{s-1} \Rightarrow y = \frac{1}{2}t^2 - \alpha + \alpha e^t \Rightarrow$
$\frac{1}{2} = \frac{1}{2} - \alpha + \alpha e \iff \alpha = 0$, cioè $y = \frac{1}{2}t^2$.

4) $\int_{-\infty}^{+\infty} \frac{e^{-i\omega t}}{t^4+1} \qquad \int_{\Gamma_R} \frac{e^{-i\omega z}}{z^4+1}\,dz = 2\pi i\left(R(f, z_2) + R(f, z_3)\right) \qquad \mathrm{Re}(-i\omega z) < 0 \iff \omega y < 0 \iff y < 0$



$\underbrace{-\int_{-R}^{R} \frac{e^{-i\omega t}}{t^4+1}\,dt}_{\downarrow R\to+\infty \quad -\int_{-\infty}^{+\infty} \frac{e^{-i\omega t}}{t^4+1}\,dt} + \underbrace{\int_{\gamma_R} \frac{e^{-i\omega z}}{z^4+1}\,dz}_{\downarrow \text{Jordan} \quad 0} =$

$z_2^3 = z_3 \quad z_3^3 = z_2$

$2\pi i\left(\frac{e^{-i\omega z_2}}{4z_2^3} + \frac{e^{-i\omega z_3}}{4z_3^3}\right) = 2\pi i\left(\frac{e^{-i\omega z_2}}{4z_3} + \frac{e^{-i\omega z_3}}{4z_2}\right)$

$= \frac{1}{2}\pi i\left(\frac{e^{-i\omega\left(-\frac{1}{\sqrt2} - \frac{1}{\sqrt2}i\right)}}{+\frac{1}{\sqrt2} - \frac{1}{\sqrt2}i} + \frac{e^{-i\omega\left(+\frac{1}{\sqrt2} - \frac{1}{\sqrt2}i\right)}}{-\frac{1}{\sqrt2} - \frac{1}{\sqrt2}i}\right) =$

$= \frac{\pi i}{2}\, e^{-\frac{\omega}{\sqrt2}}\left[\left(-\frac{1}{\sqrt2} - \frac{1}{\sqrt2}i\right)e^{i\frac{\omega}{\sqrt2}} + \left(\frac{1}{\sqrt2} - \frac{1}{\sqrt2}i\right)e^{-i\frac{\omega}{\sqrt2}}\right] =$

$= \frac{\pi i}{\sqrt2}\,\frac{e^{-\frac{\omega}{\sqrt2}}}{2}\left(-\sin\left(\frac{\omega}{\sqrt2}\right)2i - \cos\left(\frac{\omega}{\sqrt2}\right)2i\right) =$

$= \frac{\pi}{\sqrt2}\, e^{-\frac{\omega}{\sqrt2}}\left(\sin\frac{\omega}{\sqrt2} + \cos\frac{\omega}{\sqrt2}\right) \Rightarrow \int_{-\infty}^{+\infty} \frac{e^{-i\omega t}}{t^4+1}\,dt = \frac{\pi}{\sqrt2}\, e^{-\frac{\omega}{\sqrt2}}\left(\sin\frac{\omega}{\sqrt2} + \cos\frac{\omega}{\sqrt2}\right)$

5) $\int_{-\pi}^{\pi} \frac{\sin^2\vartheta}{1 + b\cos\vartheta}\,d\vartheta = \int_{C_1(0)} \frac{\left(\frac{z - z^{-1}}{2i}\right)^2}{1 + b\,\frac{z + z^{-1}}{2}}\,\frac{1}{zi}\,dz = \int_{C_1(0)} \frac{\left(\frac{z^2-1}{2zi}\right)^2 2z}{2z + bz^2 + b}\,\frac{1}{zi}\,dz$

$= -\int_{C_1(0)} \frac{z^2 - 1}{2b\left(z - \frac{-1+\sqrt{1-b^2}}{b}\right)\left(z - \frac{-1-\sqrt{1-b^2}}{b}\right)} \ldots i, =$

$$-2\pi i\left[R\left(f, 0\right) + R\left(f, \frac{-1+\sqrt{1-b^2}}{b}\right)\right] =$$

$$-2\pi i\left[-\frac{1}{b} + \left(\left(\frac{-1+\sqrt{1-b^2}}{b}\right)^2 - 1\right) \Big/ \left(\left(-1+\sqrt{1-b^2}\right) 2\, \frac{\sqrt{1-b^2}}{b}\right)\right] = 0$$