1/1

Analisi matematica IV - TLC 2/2/09

1) Assegnato $y \in \mathbb{R}^3 : \|y\| < 1$, individuare la funzione $f : x \in \mathbb{R}^3 - \{y\} \to f(x) \in]0, +\infty[$ tale che

$$\| y + f(x)(x-y) \| = 1 \quad \forall x \in \mathbb{R}^3 - \{y\}.$$

2) Si ricostruisca la funzione razionale $g(z)$ sapendo che

a) $g(0) = 1$, b) g ha solo due poli, uno semplice in $z = -1$, con residuo 1, ed uno doppio in $z = 1$, con residuo 1, c) $\lim_{z \to \infty} g(z) = 2$

3) Usando la trasformata di Laplace, risolvere il seguente problema:

$$\begin{cases} y'' - y' = 1 - t \\ y(0) = 0 \\ y(1) = \frac{1}{2} \end{cases}$$

4) Provare che

$$\mathcal{F}\left[\frac{1}{x^4+1}\right](\omega) = \frac{\pi}{\sqrt{2}} e^{-\frac{\omega}{\sqrt{2}}} \left(\cos\left(\frac{\omega}{\sqrt{2}}\right) + \operatorname{sen} \frac{\omega}{\sqrt{2}} \right) \quad \forall \omega > 0$$

(Usare il lemma di Jordan).

5) Assegnato $b \in]0, 1[$, ~~calcolare~~ dimostrare che

$$\int_{-\pi}^{\pi} \frac{\operatorname{sen} \vartheta}{1 + b \cos \vartheta} \, d\vartheta = 0,$$

usando tecniche di analisi complessa.

1) $f(x)\ \exists!\ t\quad \|y+t(x-y)\|^2=1 \iff \|y\|^2+t^2\|x-y\|^2+2ty(x-y)=1 \iff$

$t^2\|x-y\|^2+2y(x-y)t+\|y\|^2-1=0 \iff t=\dfrac{-y(x-y)+\sqrt{(y(x-y))^2-(\|y\|^2-1)\|x-y\|^2}}{\|x-y\|^2}$

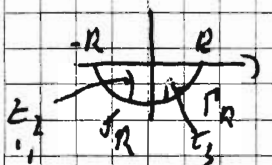
2) $y(t)=\frac{1}{t+1}+\frac{1}{t-1}+\frac{\alpha}{(t-1)^2}+2,\quad y(0)=1 \Rightarrow 1-1+\alpha+2=1 \iff \alpha=-1$

3) $s^2\mathcal{L}[y]-\alpha-s\mathcal{L}[y]=\frac{1}{s}-\frac{1}{s^2} \iff \mathcal{L}[y]=\left(\frac{1}{s}-\frac{1}{s^2}+\alpha\right)\frac{1}{s^2-s}=\frac{s-1+s^2\alpha}{s^2}\,\frac{1}{s^2-s}=$

$=\frac{1}{s^3}+\frac{\alpha}{s(s-1)}=\frac{1}{s^3}-\frac{\alpha}{s}+\frac{\alpha}{s-1} \Rightarrow y=\frac{1}{2}t^2-\alpha+\alpha e^t \Rightarrow$

$\frac{1}{2}=\frac{1}{2}-\alpha+\alpha e \iff \alpha=0$, cioè $y=\frac{1}{2}t^2$.

4) $\displaystyle\int_{-\infty}^{+\infty}\frac{e^{-i\omega t}}{t^4+1}\qquad \int_{\Gamma_R}\frac{e^{-i\omega z}}{z^4+1}dz=2\pi i\left(R(f,z_2)+R(f,z_3)\right)$ $\quad Re(-i\omega z)<0 \iff \omega y<0 \iff y<0$



$-\frac{1}{\sqrt2}-\frac{1}{\sqrt2}i \qquad \frac{1}{\sqrt2}-\frac{1}{\sqrt2}i$

$\displaystyle -\int_{-R}^{R}\frac{e^{-i\omega t}}{t^4+1}dt+\int_{\gamma_R}\frac{e^{-i\omega z}}{z^4+1}dz$ ''

$\downarrow R\to+\infty \qquad \downarrow$ Jordan

$\displaystyle -\int_{-\infty}^{+\infty}\frac{e^{-i\omega t}}{t^4+1}dt \qquad 0$

$z_2^3=z_3 \quad z_3^3=z_2$

$\hookrightarrow 2\pi i\left(\dfrac{e^{-i\omega z_2}}{4z_2^3}+\dfrac{e^{-i\omega z_3}}{4z_3^3}\right)=2\pi i\left(\dfrac{e^{-i\omega z_2}}{4z_3}+\dfrac{e^{-i\omega z_3}}{4z_2}\right)$

$=\frac{1}{2}\pi i\left(\dfrac{e^{-i\omega\left(-\frac{1}{\sqrt2}-\frac{1}{\sqrt2}i\right)}}{+\frac{1}{\sqrt2}-\frac{1}{\sqrt2}i}+\dfrac{e^{-i\omega\left(+\frac{1}{\sqrt2}-\frac{1}{\sqrt2}i\right)}}{-\frac{1}{\sqrt2}-\frac{1}{\sqrt2}i}\right)=$

$=\frac{\pi i}{2}e^{-\frac{\omega}{\sqrt2}}\left[\left(-\frac{1}{\sqrt2}-\frac{1}{\sqrt2}i\right)e^{i\frac{\omega}{\sqrt2}}+\left(\frac{1}{\sqrt2}-\frac{1}{\sqrt2}i\right)e^{-i\frac{\omega}{\sqrt2}}\right]=$

$=\frac{\pi i}{\sqrt2}\,\frac{e^{-\frac{\omega}{\sqrt2}}}{2}\left(-\mathrm{sen}\left(\frac{\omega}{\sqrt2}\right)2i-\cos\left(\frac{\omega}{\sqrt2}\right)2i\right)=$

$=\frac{\pi}{\sqrt2}e^{-\frac{\omega}{\sqrt2}}\left(\mathrm{sen}\frac{\omega}{\sqrt2}+\cos\frac{\omega}{\sqrt2}\right) \Rightarrow \displaystyle\int_{-\infty}^{+\infty}\frac{e^{-i\omega t}}{t^4+1}dt=\frac{\pi}{\sqrt2}e^{-\frac{\omega}{\sqrt2}}\left(\mathrm{sen}\frac{\omega}{\sqrt2}+\cos\frac{\omega}{\sqrt2}\right)$

5) $\displaystyle\int_{-\pi}^{\pi}\frac{\mathrm{sen}\,\vartheta}{1+b\cos\vartheta}d\vartheta=\int_{C_1(0)}\frac{\frac{z-z^{-1}}{2i}}{1+b\frac{z+z^{-1}}{2}}\,\frac{1}{zi}\,dz=\int_{C_1(0)}\frac{\frac{z^2-1}{2zi}}{\frac{2z+bz^2+b}{2z}}\,\frac{1}{zi}\,dz$

$\displaystyle=-\int_{C_1(0)}\frac{z^2-1}{z\,b\left(z-\frac{-1+\sqrt{1-b^2}}{b}\right)\left(z-\frac{-1-\sqrt{1-b^2}}{b}\right)}\,dz=$

$$-2\pi i\left[R(f,0)+R\left(f,\frac{-1+\sqrt{1-b^2}}{b}\right)\right]=$$

$$-2\pi i\left[-\frac{1}{b}+\left(\left(\frac{-1+\sqrt{1-b^2}}{b}\right)^2-1\right)\Big/\left(\left(-1+\sqrt{1-b^2}\right)2\,\frac{\sqrt{1-b^2}}{b}\right)\right]=0$$