Quasi-static approximations of Maxwell equations

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1. General considerations

We focus our attention on the "curl" type Maxwell equations in the frequency domain:

$$\nabla \times \mathbf{E} = -j\omega \mathbf{B} \tag{1}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + j\omega \mathbf{D} \tag{2}$$

where standard symbols have been used: $\mathbf{E}(\mathbf{x})$ is the electric field, $\mathbf{B}(\mathbf{x})$ the magnetic flux density, $\mathbf{H}(\mathbf{x})$ the magnetic field, $\mathbf{D}(\mathbf{x})$ the electric displacement, $\mathbf{J}(\mathbf{x})$ the current density, and $\mathbf{x}$ is the spatial coordinate.
To close the problem, the "div" type equations must be added, together with suitable constitutive relations, and boundary conditions, e.g. related to external sources (currents or charges). We assume to deal with linear materials, although this assumption is not strictly necessary, as we will discuss in the following, so that:

$$\begin{aligned} \mathbf{D} &= \varepsilon(\mathbf{x})\mathbf{E} \\ \mathbf{B} &= \mu(\mathbf{x})\mathbf{H} \\ \mathbf{J} &= \sigma(\mathbf{x})\mathbf{E} \end{aligned} \tag{3}$$

Now, we introduce some reference quantities E, B, H, D, J, L for the electromagnetic fields and for the spatial coordinate $\mathbf{x}$, so that:

$$\begin{cases} \mathbf{x} = L\,\boldsymbol{\xi} \\ \mathbf{E}(\mathbf{x}) = E\ \mathbf{e}(\boldsymbol{\xi}) \\ \mathbf{B}(\mathbf{x}) = B\ \mathbf{b}(\boldsymbol{\xi}) \\ \mathbf{H}(\mathbf{x}) = H\ \mathbf{h}(\boldsymbol{\xi}) \\ \mathbf{D}(\mathbf{x}) = D\ \mathbf{d}(\boldsymbol{\xi}) \\ \mathbf{J}(\mathbf{x}) = J\ \mathbf{j}(\boldsymbol{\xi}) \end{cases} \tag{4}$$

The reference quantities are chosen so that the non-dimensional quantities $\boldsymbol{\xi}$, $\mathbf{e}$, $\mathbf{b}$, $\mathbf{h}$, $\mathbf{d}$, $\mathbf{j}$ are of order 1. This can be achieved by choosing as reference quantities the root mean square values of the related electromagnetic quantity, and supposing that the electromagnetic fields cannot be very different from such mean values. The length L will be a typical length of the system considered. Of course, it may be necessary to subdivide the original solution domain in sub-domains over which the electromagnetic quantities are not dramatically varying. What we are going to say will be then valid for one of such sub-domains.

We notice immediately that, due to (3), the quantities D, J, H depend on E, B, since for instance:

$$\mathbf{D} = \varepsilon(\mathbf{x})\mathbf{E} \quad \Rightarrow \quad \begin{cases} \mathbf{d} = \varepsilon_{norm}(\boldsymbol{\xi})\ \mathbf{e} \\ D = \varepsilon_{ref}\, E \end{cases} \tag{5}$$

where ε_{ref} is a (constant) reference value for the electric permittivity such that the normalized value $\varepsilon_{norm} = \varepsilon/\varepsilon_{ref}$ is of order 1. One possible choice could be $\varepsilon_{ref} = \varepsilon_0$ (vacuum permittivity), so that ε_{norm}

= ε_r (relative permittivity), if ε_r is of order 1. Notice that in fact it is not necessary that the constitutive relation is linear; what we require is that it is possible to define ε_{ref} that relates D and E, such that **d** is of order 1 when **e** is of order 1.
Similarly, we have:

$$\mathbf{B} = \mu(\mathbf{x})\mathbf{H} \quad \Rightarrow \quad \begin{cases} \mathbf{b} = \mu_{norm}(\xi)\, \mathbf{h} \\ B = \mu_{ref}\, H \end{cases} \tag{6}$$

$$\mathbf{J} = \sigma(\mathbf{x})\mathbf{E} \quad \Rightarrow \quad \begin{cases} \mathbf{j} = \sigma_{norm}(\xi)\, \mathbf{e} \\ J = \sigma_{ref}\, E \end{cases} \tag{7}$$

Rewriting (1) in view of (4) we have:

$$\nabla_\xi \times \mathbf{e} = -j\omega \frac{B}{E} L\, \mathbf{b} \tag{8}$$

We define:

$$\begin{cases} \tau_{em} = \dfrac{L}{c} \\ \alpha = c\dfrac{B}{E} \\ c = \dfrac{1}{\sqrt{\varepsilon_{ref}\, \mu_{ref}}} \end{cases} \tag{9}$$

i.e. c is the speed of electromagnetic waves in the material under consideration, τ_{em} is the time needed to electromagnetic waves to travel the typical length L, α is linked to the squared ratio of magnetic and electric energies:

$$\alpha = c\frac{B}{E} = \frac{1}{\sqrt{\varepsilon_{ref}\, \mu_{ref}}} \frac{\sqrt{\frac{1}{V}\int_V |\mathbf{B}|^2\, dV}}{\sqrt{\frac{1}{V}\int_V |\mathbf{E}|^2\, dV}} = \sqrt{\frac{\int_V \frac{|\mathbf{B}|^2}{2\mu_{ref}}\, dV}{\int_V \frac{1}{2}\varepsilon_{ref} |\mathbf{E}|^2\, dV}} \tag{10}$$

With these definitions, (8) becomes

$$\nabla_\xi \times \mathbf{e} = -j\omega\, \tau_{em}\, \alpha\, \mathbf{b} \tag{11}$$

Using similar arguments on (2) we have:

$$\nabla_\xi \times \mathbf{h} = \frac{LJ}{H}\mathbf{j} + j\omega \frac{D}{H} L\, \mathbf{d} \tag{12}$$

that becomes, using (5)-(7):

$$\nabla_\xi \times \mathbf{h} = L\, \mu_{ref}\, \sigma_{ref} \frac{E}{B}\mathbf{j} + j\omega\, L\, \mu_{ref}\, \varepsilon_{ref} \frac{E}{B}\mathbf{d} \tag{13}$$

Now, we introduce two well-known characteristic electromagnetic times:

$$\begin{cases} \tau_m = \mu_{ref}\, \sigma_{ref} L^2 \\ \tau_e = \dfrac{\varepsilon_{ref}}{\sigma_{ref}} \end{cases} \tag{14}$$

The time τ_e is the electric charge diffusion time, i.e. the characteristic time with which the unpaired electric charge decays in a conductor. The time τ_m is the current density diffusion time, i.e. the characteristic time with which the current density (and hence the magnetic field) penetrates in a conductor.

It is easily seen that it results:

$$\tau_{em}^2 = \tau_m \, \tau_e \tag{15}$$

and that

$$L \, \mu_{ref} \, \sigma_{ref} \, c = \frac{\tau_m}{\tau_{em}} = \frac{\tau_{em}}{\tau_e} \tag{16}$$

so that (12) becomes:

$$\nabla_\xi \times \mathbf{h} = \frac{\tau_m}{\tau_{em}} \frac{1}{\alpha} \mathbf{j} + j\omega \, \tau_{em} \frac{1}{\alpha} \mathbf{d} \tag{17a}$$

or

$$\nabla_\xi \times \mathbf{h} = \frac{\tau_{em}}{\tau_e} \frac{1}{\alpha} \mathbf{j} + j\omega \, \tau_{em} \frac{1}{\alpha} \mathbf{d} \tag{17b}$$

In the end, the equations to be considered are (11) and (17) (a or b).

We want to answer to the following question: “Which are the equations to be solved in the low frequency limit?”, where “low frequency” means obviously:

$$\omega \, \tau_{em} << 1 \tag{18a}$$

We can quantify the “much smaller” in (18a), deciding a threshold value k<1 such that (18a) means in fact:

$$\omega \, \tau_{em} < k \tag{18b}$$

A possible choice could be, for instance, k = 0.1.

The answer about the equations to be solved in the low frequency limit is not trivial, because the term $\omega\tau_{em}$ appears together with other terms like α and the other electromagnetic times. Moreover, the term α in general varies with frequency; its behaviour depends also on the particular geometry of the region under study. We have three possibilities, corresponding to the fact that the order of magnitude of α can vary so that one of the terms in which it is present is of order 1:

$$\text{Case 1: } \alpha \approx \frac{1}{\omega \, \tau_{em}} \tag{19a}$$

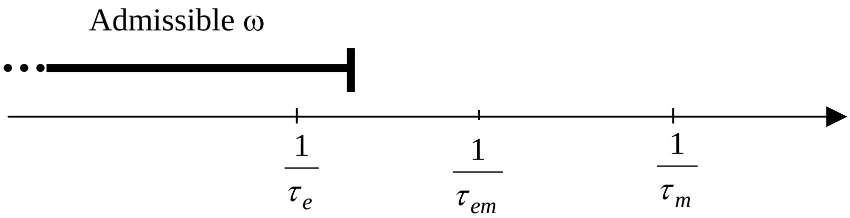
$$\text{Case 2: } \alpha \approx \omega \, \tau_{em} \tag{19b}$$

$$\text{Case 3: } \alpha \approx \frac{\tau_m}{\tau_{em}} = \frac{\tau_{em}}{\tau_e} \tag{19c}$$

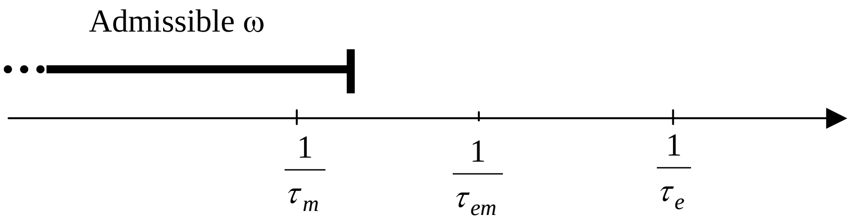
where”≈” stands for “of the order of”. To check the occurrence of one of such cases, it is sufficient to examine α in the limit $\omega\tau_{em} \to 0$, i.e. in the static limit. In case 1, we have that the energy related to the electric field goes to zero, that means that current flows in perfect conductors. In case 2, the energy related to the magnetic field goes to zero, that means that no current is present, that is the electric field is present in some perfect insulators. In case 3 no perfect materials are present; this is evidently the most realistic situation. In this last case α can well be much greater or lower than 1, depending on the various situations; what we are saying is that the order of magnitude of α does not scale with frequency. We concentrate on the latter case.

However, we must further distinguish three possibilities, that are depicted schematically in the two figures. Notice that τ_{em} is always in between τ_e and τ_m, thanks to (15).

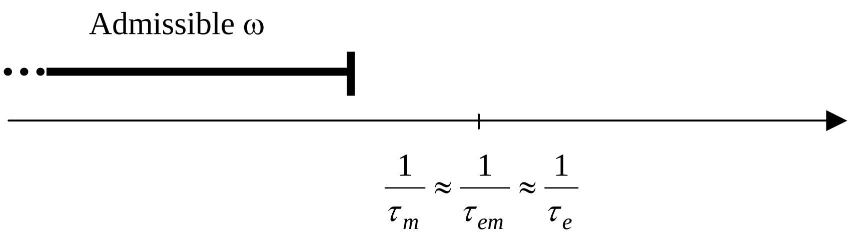
CASE A: $\frac{1}{\tau_e} << \frac{1}{\tau_{em}} << \frac{1}{\tau_m} \quad \Rightarrow \quad \frac{1}{\tau_e} < k\frac{1}{\tau_{em}}, \quad \frac{1}{\tau_{em}} < k\frac{1}{\tau_m}$



CASE B: $\frac{1}{\tau_m} << \frac{1}{\tau_{em}} << \frac{1}{\tau_e} \quad \Rightarrow \quad \frac{1}{\tau_m} < k\frac{1}{\tau_{em}}, \quad \frac{1}{\tau_{em}} < k\frac{1}{\tau_e}$



CASE C: $\frac{1}{\tau_m} \approx \frac{1}{\tau_{em}} \approx \frac{1}{\tau_e}$



Let us recall the equations to be solved:

$$\nabla_\xi \times \mathbf{e} = -j\omega\, \tau_{em}\, \alpha\, \mathbf{b} \tag{20}$$

$$\nabla_\xi \times \mathbf{h} = \frac{\tau_{em}}{\tau_e}\frac{1}{\alpha}\mathbf{j} + j\omega\, \tau_{em}\, \frac{1}{\alpha}\mathbf{d} \tag{21}$$

where, thanks to (19c):

$$\omega\, \alpha\, \tau_{em} \approx \omega\, \tau_m \tag{22}$$

and

$$\omega\, \tau_{em}\frac{1}{\alpha} \approx \omega\, \tau_e \tag{23}$$

CASE A

In case A, it results, from (19c), that α<<1; this means that the energy of the electric field is dominant over the energy of the magnetic field. Moreover, we have:

$$\omega\, \tau_m < \; k\, \omega\, \tau_{em} < k^2 \tag{24}$$

We decide to neglect the terms of order k^2, so that the right hand side of (39) is negligible:

$$\nabla_\xi \times \mathbf{e} \cong 0 \tag{25}$$

We can distinguish various different possibilities.

- "Relatively high frequencies", i.e. $\omega \approx \dfrac{1}{\tau_e}$

In this case also the second term in the right hand side of (21) is of order 1, so that the equations to be solved are:

$$\begin{cases} \nabla \times \mathbf{E} = 0 \\ \nabla \times \mathbf{H} = \mathbf{J} + j\,\omega\,\mathbf{D} \end{cases} \tag{26}$$

that are the *ElectroQuasiStatic* equations (EQS). If an equivalent circuit is looked for, evidently in this case we will obtain a number of capacitors and resistors.

- "Extremely low frequencies", i.e. $\omega \approx k^2\dfrac{1}{\tau_e}$ (note the k^2)

In this case we have:

$$\omega\, \tau_e \approx \; k^2 \tag{27}$$

so that, neglecting k^2, we have:

$$\nabla_\xi \times \mathbf{h} \cong \frac{\tau_{em}}{\tau_e}\frac{1}{\alpha}\mathbf{j} \tag{28}$$

and the equations to be solved are:

$$\begin{cases} \nabla \times \mathbf{E} = 0 \\ \nabla \times \mathbf{H} = \mathbf{J} \end{cases} \tag{29}$$

that are the equations of the *Quasi Stationary Conduction* (QSC).
If an equivalent circuit is looked for, evidently in this case we will obtain a number of resistors alone.

- "Low frequencies", i.e. $\omega \approx k \dfrac{1}{\tau_e}$

In this case we have:

$$\omega\, \tau_e \approx k \tag{30}$$

This situation is borderline. Indeed, if we decide to neglect also terms of the order of *k*, then we have QSC again; if we retain such terms we have EQS again.

CASE B

In case B, it results, from (19c), that α>>1; this means that the energy of the magnetic field is dominant over the energy of the electric field. Moreover, we have:

$$\omega\, \tau_e < k\; \omega\, \tau_{em} < k^2 \tag{31}$$

and hence, neglecting terms of order k^2, (21) becomes:

$$\nabla_\xi \times \mathbf{h} \cong \frac{\tau_{em}}{\tau_e} \frac{1}{\alpha} \mathbf{j} \tag{32}$$

In other words, the displacement current is negligible.

Hence, we can distinguish various different possibilities.

- "Relatively high frequencies", i.e. $\omega \approx \dfrac{1}{\tau_m}$

In this case the right hand side of (20) is of order 1, so that the equations to be solved are:

$$\begin{cases} \nabla \times \mathbf{E} = -j\, \omega\, \mathbf{B} \\ \nabla \times \mathbf{H} = \mathbf{J} \end{cases} \tag{33}$$

that are the *MagnetoQuasiStatic* equations (MQS).
If an equivalent circuit is looked for, evidently in this case we will obtain a number of inductors and resistors.

- "Extremely low frequencies", i.e. $\omega \approx k^2 \dfrac{1}{\tau_m}$ (note the k^2)

In this case we have:

$$\omega\, \tau_m \approx k^2 \tag{34}$$

so that, neglecting k^2, we have from (20):

$$\nabla_\xi \times \mathbf{e} \cong 0 \tag{35}$$

so that the limiting equations are:

$$\begin{cases} \nabla \times \mathbf{E} = 0 \\ \nabla \times \mathbf{H} = \mathbf{J} \end{cases} \tag{36}$$

that are the equations of the *Quasi Stationary Conduction* (QSC).
If an equivalent circuit is looked for, evidently in this case we will obtain a number of resistors.

- "Low frequencies", i.e. $\omega \approx k \dfrac{1}{\tau_m}$

In this case we have:

$$\omega \tau_m \approx k \tag{37}$$

This situation is borderline. Indeed, if we decide to neglect also terms of the order of k, then we have QSC again; if we retain such terms we have MQS again.

CASE C

In case C, it results, from (19c), that $\alpha \approx 1$; this means that the energy of the magnetic field is of the same order of magnitude of the energy of the electric field. Moreover, we have:

$$\omega \tau_e \approx \omega \tau_{em} \approx \omega \tau_m \tag{38}$$

i.e. the right hand side of (20) and the second term of the right hand side of (21) are of the same order.
We have two possibilities (remember (18)):

- "Extremely low frequencies", i.e. $\omega \approx k^2 \dfrac{1}{\tau_m}$ (note the k^2)

In this case we have:

$$\omega \tau_m \approx \omega \tau_e \approx k^2 \tag{39}$$

and hence the two terms are both negligible, and the situation is QSC.

- "Low frequencies", i.e. $\omega \approx k \dfrac{1}{\tau_m}$

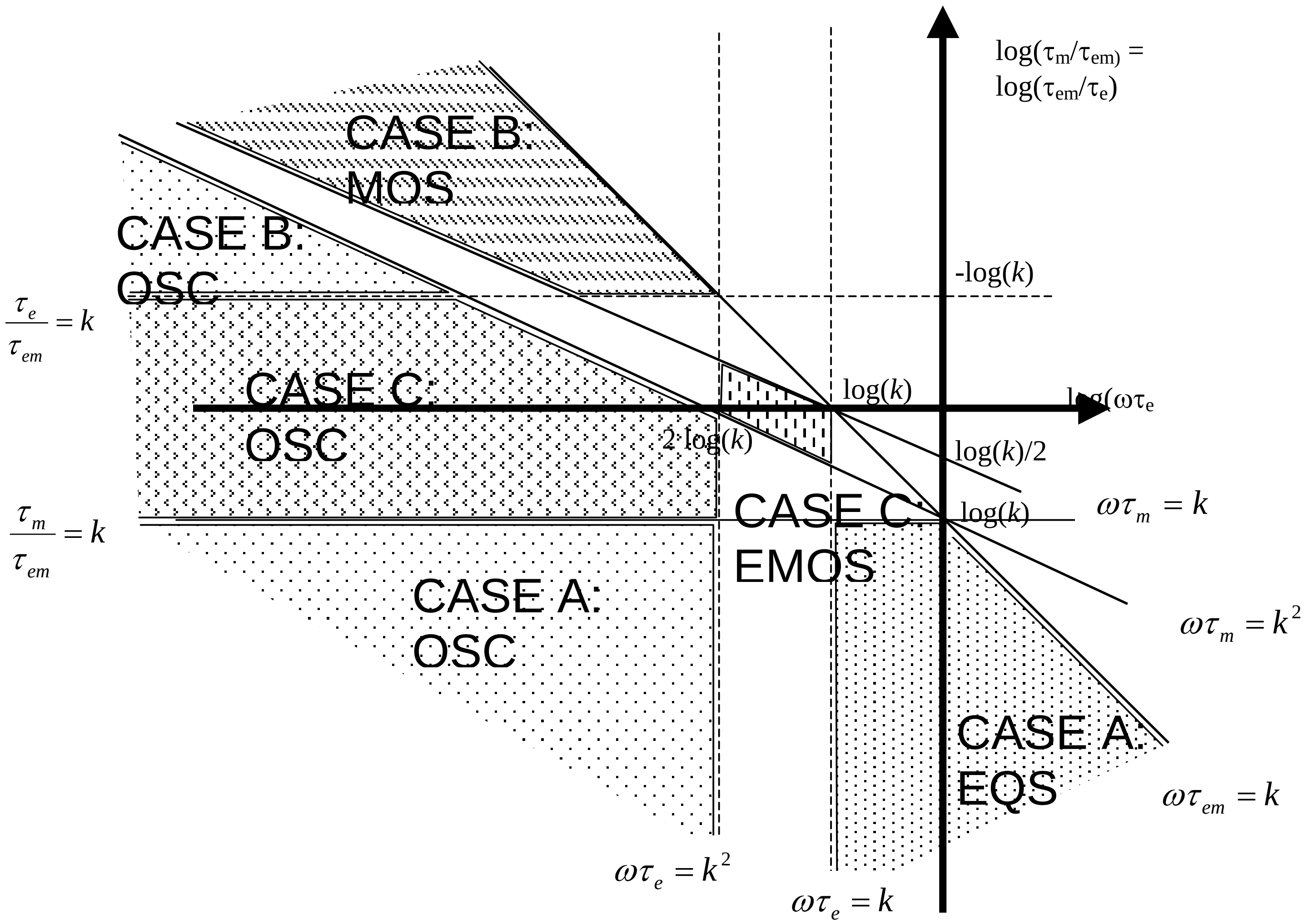
In this case we have:

$$\omega \tau_e \approx \omega \tau_m \approx k \tag{40}$$

If we decide to neglect also terms of the order of k, then we have QSC again. Conversely, if we retain such terms we have:

$$\begin{cases} \nabla \times \mathbf{E} = -j\,\omega\,\mathbf{B} \\ \nabla \times \mathbf{H} = \mathbf{J} + j\,\omega\,\mathbf{D} \end{cases} \tag{41}$$

that are the full Maxwell equations, but always in the low frequency limit (18). This implies that propagation may be still negligible, so that we can call this situation as *ElectroMagnetic Quasi Static* (EMQS).
If an equivalent circuit is looked for, evidently in this case we will obtain a number of capacitors, inductors and resistors.


log(τm/τem) = log(τem/τe)
-log(k)
log(k)
log(ωτe
2·log(k)
log(k)/2
log(k)
CASE B: MOS
CASE B: OSC
CASE C: OSC
CASE C: EMOS
CASE A: OSC
CASE A: EQS
$\frac{\tau_e}{\tau_{em}} = k$
$\frac{\tau_m}{\tau_{em}} = k$
$\omega\tau_m = k$
$\omega\tau_m = k^2$
$\omega\tau_{em} = k$
$\omega\tau_e = k^2$
$\omega\tau_e = k$

2. An example of application

We consider a test case depicted in Fig. 1: a linear material with uniform σ, μ and ε between two perfectly conducting plane electrodes, fed with a sinusoidal voltage generator $\overline{V}_g$ with angular frequency ω at one end. A one-port element is connected at the other end.

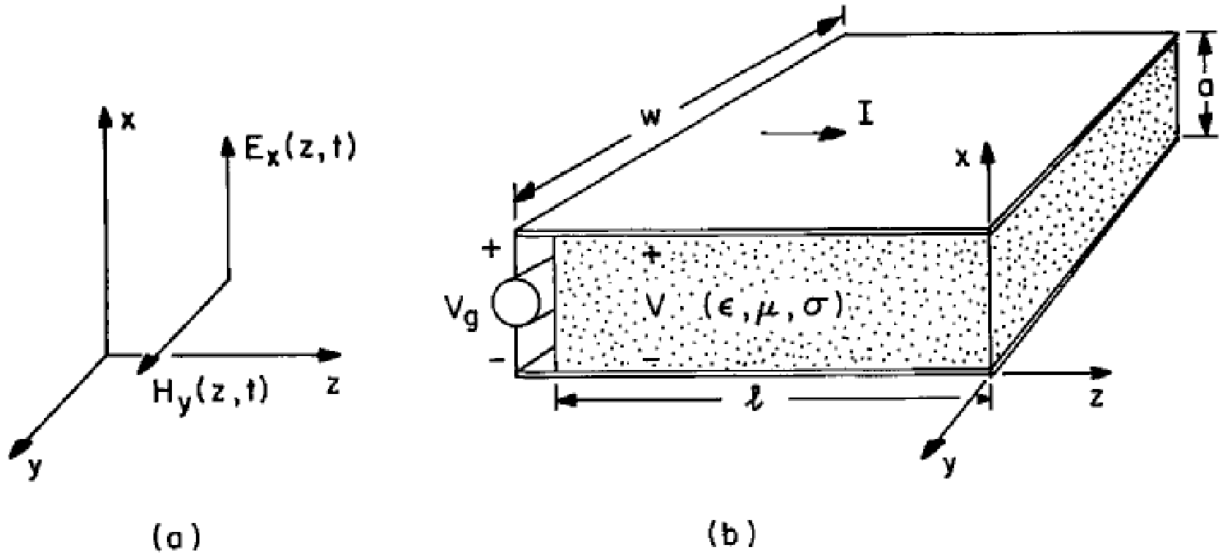


Fig. 1. The test case (from [1]).

Ignoring fringing effects, the electromagnetic fields are as follows:

$$\mathbf{E}=\overline{E}(z)\hat{\mathbf{i}}_x, \mathbf{D}=\overline{D}(z)\hat{\mathbf{i}}_x, \mathbf{J}=\overline{J}(z)\hat{\mathbf{i}}_x \qquad (42)$$
$$\mathbf{H}=\overline{H}(z)\hat{\mathbf{i}}_y, \mathbf{B}=\overline{B}(z)\hat{\mathbf{i}}_y$$

where $\overline{E}(z)$ and $\overline{H}(z)$ are suitable phasors depending on the spatial coordinate z. The time domain behaviour of the various quantities can be recovered as, for instance, $E(z,t)=\mathrm{Im}\left(\overline{E}(z)\right)e^{j\omega t}$.

The equations to be solved become:

$$\frac{d\overline{E}}{dz}=-j\omega\overline{B} \qquad (43a)$$

$$-\frac{d\overline{H}}{dz}=\overline{J}+j\omega\overline{D} \qquad (43b)$$

with the following constitutive equations :

$$\overline{D}=\varepsilon\,\overline{E}$$
$$\overline{B}=\mu\,\overline{H} \qquad (44)$$
$$\overline{J}=\sigma\,\overline{E}$$

The boundary conditions are:

$$\bar{E}(z=-l) = -\frac{\bar{V}_g}{a} \tag{45a}$$

$$\bar{E}(z=0) = -\frac{\bar{V}_e}{a} \tag{45b}$$

$$\bar{H}(z=0) = \frac{\bar{I}_e}{w} \tag{45c}$$

$$\bar{H}(z=-l) = \frac{\bar{I}_g}{w} \tag{45d}$$

where V_e, Ie are the voltage and the current at the end of the plate, and V_g and I_g are the voltage and the current at the generator. Assuming a sinusoidal voltage generator and that at the end an open circuit is connected, then V_g is known and I_e=0, while V_e and I_g are to be determined.

The solution of the "full" equations (43a) can be done as follows. Due to the symmetry of the electromagnetic field, a voltage between the plates and a current flowing into the plates can be defined at each z coordinate as follows:

$$\bar{V}(z) = -a\,\bar{E}(z) \tag{46a}$$

$$\bar{I}(z) = \bar{H}(z)\,w \tag{46b}$$

Using (46) and combining (43) and (44) we obtain the following equations:

$$\frac{d\bar{V}}{dz} = -j\omega L\bar{I} \tag{47a}$$

$$\frac{d\bar{I}}{dz} = -(G + j\omega C)\bar{V} \tag{47b}$$

where

$$G = \frac{\sigma w}{a},\quad C = \frac{\varepsilon w}{a},\quad L = \frac{\mu a}{w} \tag{48}$$

These are the standard transmission line equations, to be solved with the following boundary conditions:

$$\bar{V}(-l) = \bar{V}_g \tag{49a}$$

$$\bar{I}(0) = 0 \tag{49b}$$

With standard manipulations we get:

$$\frac{d^2\bar{V}}{dz^2} - j\omega L(G + j\omega C)\bar{V} = 0 \tag{50}$$

and hence

$$\bar{V} = \bar{V}_+ e^{-j\beta z} + \bar{V}_- e^{j\beta z} \tag{51a}$$

$$\bar{I} = \frac{1}{Z_0}(\bar{V}_+ e^{-j\beta z} - \bar{V}_- e^{j\beta z}) \tag{51b}$$

with

$$\beta = \sqrt{\omega^2 LC - j\omega LG} = \sqrt{\omega^2\mu\varepsilon - j\omega\sigma\mu} = \omega\sqrt{\mu\varepsilon}\sqrt{1 - j\frac{1}{\omega\tau_e}},\quad \mathrm{Re}\,\beta > 0 \tag{52}$$

$$Z_0 = \frac{\omega L}{\beta}$$

Applying the boundary conditions we obtain:

$$\bar{V}(z) = \bar{V}_g\frac{(e^{-j\beta z} + e^{j\beta z})}{(e^{j\beta l} + e^{-j\beta l})} \tag{53a}$$

$$\bar{I}(z) = \frac{\bar{V}_g}{Z_0} \frac{(e^{-j\beta z} - e^{j\beta z})}{(e^{j\beta l} + e^{-j\beta l})} \tag{53b}$$

Now, we want to study the limiting equations in the three cases mentioned above.

<u>CASE A</u>: $\frac{1}{\tau_e} << \frac{1}{\tau_{em}} << \frac{1}{\tau_m}$

In this case, assuming a relatively high frequency, we have that the equations to be solved are the EQS ones, obtained neglecting the right hand side of (43a):

$$\frac{d\bar{E}}{dz} = 0 \tag{54a}$$

$$-\frac{d\bar{H}}{dz} = \bar{J} + j\omega\bar{D} \tag{54b}$$

From (54a) we have that

$$\bar{E}(z) = \bar{E}_0 \tag{55}$$

i.e. the electric field is independent of z, while from (54b) it results:

$$-\frac{d\bar{H}}{dz} = (\sigma + j\omega\varepsilon)\bar{E}_0 \quad \Rightarrow \quad \bar{H}(z) = -z\,(\sigma + j\omega\varepsilon)\bar{E}_0 \tag{56}$$

In order to find out an equivalent circuit, we observe that voltage and are:

$$\bar{V}_g = -a\bar{E}_0 = \bar{V}_e \tag{57a}$$

$$\bar{I}_g = \bar{H}(z=-l)\,w = l\,w\,(\sigma + j\omega\varepsilon)\bar{E}_0 \tag{57b}$$

Finally, the ratio between current and voltage is:

$$\dot{Y}_g = \frac{\bar{I}_g}{\bar{V}_g} = \frac{l\,w\,(\sigma + j\omega\varepsilon)}{a} = \frac{l\,w\,\sigma}{a} + j\omega\frac{l\,w\,\varepsilon}{a} = G\,l + j\omega C\,l \tag{58}$$

where G and C are the per unit length conductance and the capacitance of the system as defined by (48). Hence, the equivalent circuit is depicted in Fig.2.

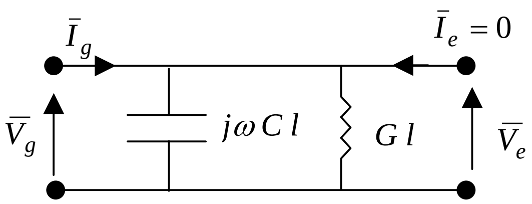
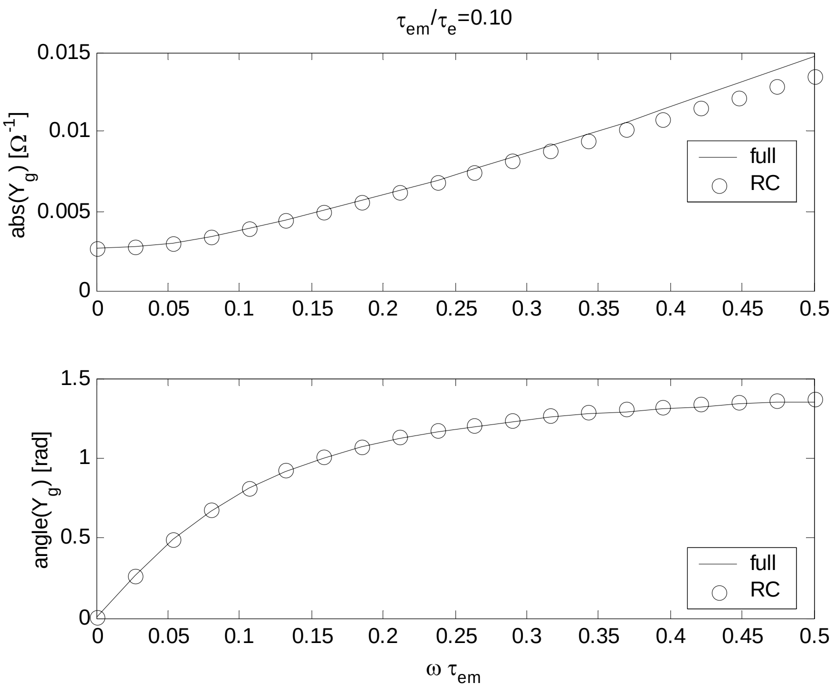
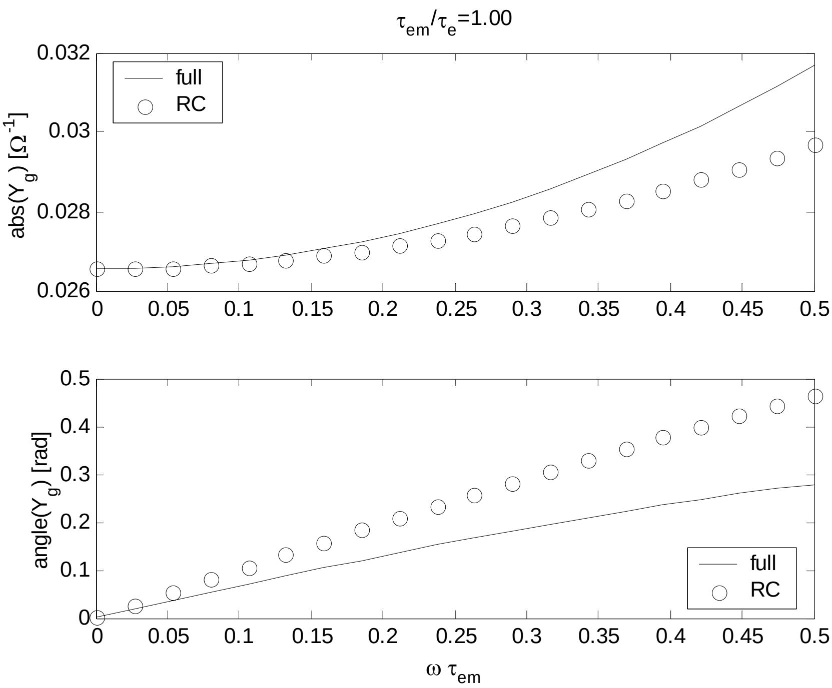


Fig. 2

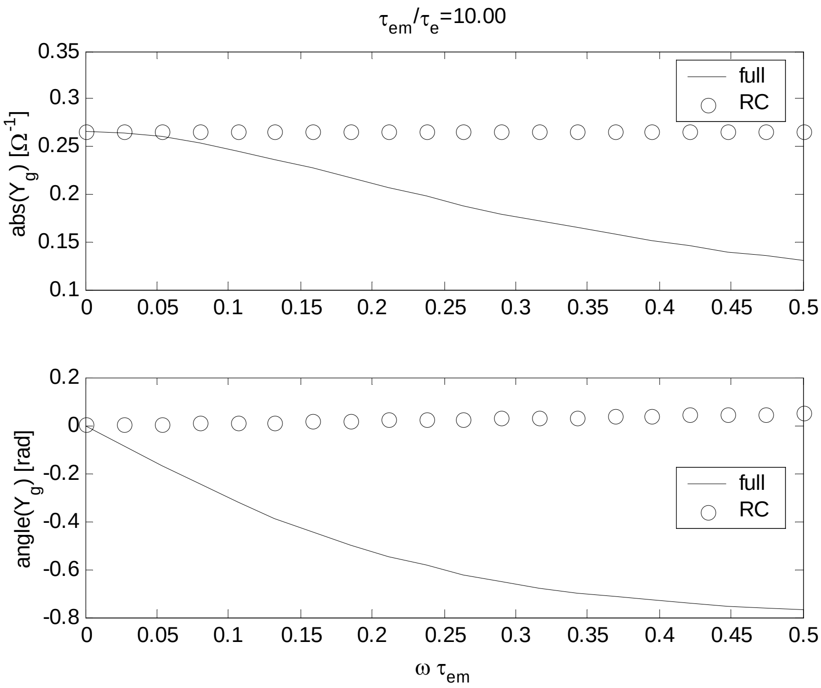
In Fig. 3 we report the behaviours of the frequency response of the circuit of Fig. 2, in terms of current, as compared with the true response given by (53b) with $z=-l$. The parameters used are: a=1cm, w=10cm, l=1m, $\varepsilon=\varepsilon_0$, $\mu=\mu_0$. The conductivity σ has been varied in order to get different ratio τ_{em}/τ_e. We can observe that when it results $\tau_{em}/\tau_e << 1$ then the equivalent circuit of fig. 2 provides the correct answer, consistently with the condition assumed to derive this model.


τ_{em}/τ_e=0.10
abs(Y_g) [Ω^{-1}]
angle(Y_g) [rad]
full
RC
$\omega\ \tau_{em}$

(a)


τ_{em}/τ_e=1.00
abs(Y_g) [Ω^{-1}]
angle(Y_g) [rad]
full
RC
$\omega\ \tau_{em}$

(b)



(c)

Fig. 3: Frequency responses

<u>CASE B</u>: $\frac{1}{\tau_m} << \frac{1}{\tau_{em}} << \frac{1}{\tau_e}$

In this case, assuming a relatively high frequency, we have that the equations to be solved are the MQS ones, obtained neglecting the displacement current:

$$\frac{d\overline{E}}{dz} = -j\omega\overline{B} \tag{59a}$$

$$-\frac{d\overline{H}}{dz} = \overline{J} \tag{59b}$$

Combining equations (59) with the constitutive relations, we get

$$\frac{d^2\overline{H}}{dz^2} - \gamma^2\overline{H} = 0 \tag{60}$$

where

$$\gamma = \frac{(1+j)}{\delta}, \quad \delta = \sqrt{\frac{2}{\omega\mu\sigma}} \tag{61}$$

and δ is the penetration depth.

The solution of (60) can be expressed as:

$$\overline{H}(z) = \overline{H}_- e^{-\gamma z} + \overline{H}_+ e^{\gamma z} \quad \Rightarrow \quad \overline{I}(z) = w\left(\overline{H}_- e^{-\gamma z} + \overline{H}_+ e^{\gamma z}\right) \tag{62}$$

and, from (59b):

$$\bar{E}(z) = \frac{\gamma}{\sigma}\left(\bar{H}_- e^{-\gamma z} - \bar{H}_+ e^{\gamma z}\right) \;\Rightarrow\; \bar{V}(z) = -a\frac{\gamma}{\sigma}\left(\bar{H}_- e^{-\gamma z} - \bar{H}_+ e^{\gamma z}\right) \tag{63}$$

The complex constants H_+, H_- must be determined with the boundary conditions.

In order to get an equivalent circuit, we find out the impedance matrix $\underline{\underline{Z}}$ of the two-port element seen at the two ends. We define:

$$\dot{Z}_{11} = \left.\frac{\bar{V}(-l)}{\bar{I}(-l)}\right|_{\bar{I}(0)=0} \tag{64a}$$

$$\dot{Z}_{21} = \left.\frac{\bar{V}(0)}{\bar{I}(-l)}\right|_{\bar{I}(0)=0} \tag{64b}$$

where $Z_{22}=Z_{11}$ for symmetry reasons and $Z_{12}=Z_{21}$ for reciprocity.
In order to find such elements, we impose $I(-l) = Ig$ and $I(0)=0$, and evaluate the voltage at the two ends. We have:

$$\bar{H}_- = -\bar{H}_+ = \frac{\bar{I}_g}{w}\frac{1}{e^{\gamma l} - e^{-\gamma l}} \tag{65}$$

and hence

$$\bar{I}(z) = \bar{I}_g \frac{e^{-\gamma z} - e^{\gamma z}}{e^{\gamma l} - e^{-\gamma l}} \tag{66a}$$

$$\bar{V}(z) = \bar{I}_g \frac{a\gamma}{w\sigma}\frac{e^{-\gamma z} + e^{\gamma z}}{e^{-\gamma l} - e^{\gamma l}} \tag{66b}$$

From (66b) it results:

$$\dot{Z}_{11} = \frac{a\gamma}{\sigma w}\frac{e^{\gamma l} + e^{-\gamma l}}{e^{-\gamma l} - e^{\gamma l}} \tag{67}$$

$$\dot{Z}_{21} = \frac{a\gamma}{\sigma w}\frac{2}{e^{-\gamma l} - e^{\gamma l}} \tag{68}$$

Now, we develop (68) in the limit $|\gamma l| << 1$, using the following Taylor series expansions:

$$e^x + e^{-x} = 2 + x^2 + O(x^4) \tag{69}$$

$$e^x - e^{-x} = 2x + O(x^3)$$

Hence, we have:

$$\dot{Z}_{11} \cong \frac{a\gamma}{\sigma w}\frac{2+\gamma^2 l^2}{2\gamma l} = \frac{a}{\sigma w l} + \frac{a l \gamma^2}{2\sigma w} = \frac{a}{\sigma w l} + j\omega\frac{\mu a}{2w}l = \frac{1}{G\,l} + j\omega\frac{L}{2}l \tag{70}$$

according to definitions (48), and

$$\dot{Z}_{21} \cong \frac{a\gamma}{\sigma w}\frac{2}{2\gamma\, l} = \frac{a}{\sigma w l} = \frac{1}{G\,l} \tag{70}$$

In the end, the equivalent circuit is the tee one depicted in Fig. 4.

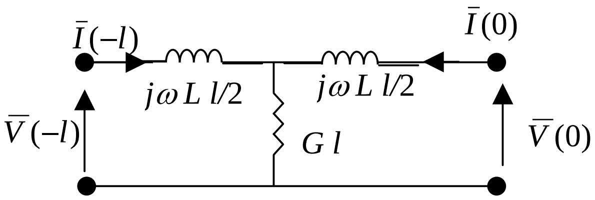


Fig. 4

Of course, in order to make more valid the approximation used to get this equivalent circuit, one could split the original system into a number of pieces of smaller length, over which the approximation $|\gamma l| << 1$ is more valid. The result would be a cascade of equivalent circuits as the one of fig. 4. In Fig. 5 we illustrate the result obtained splitting the original system in two.

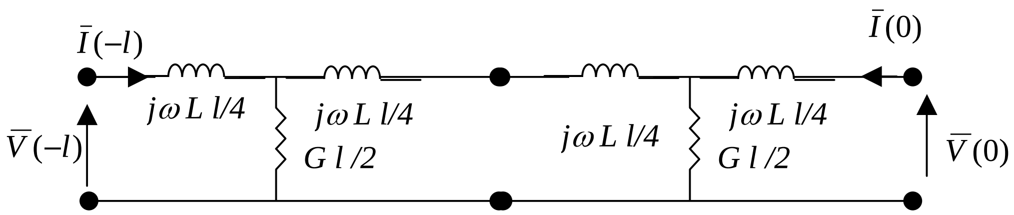
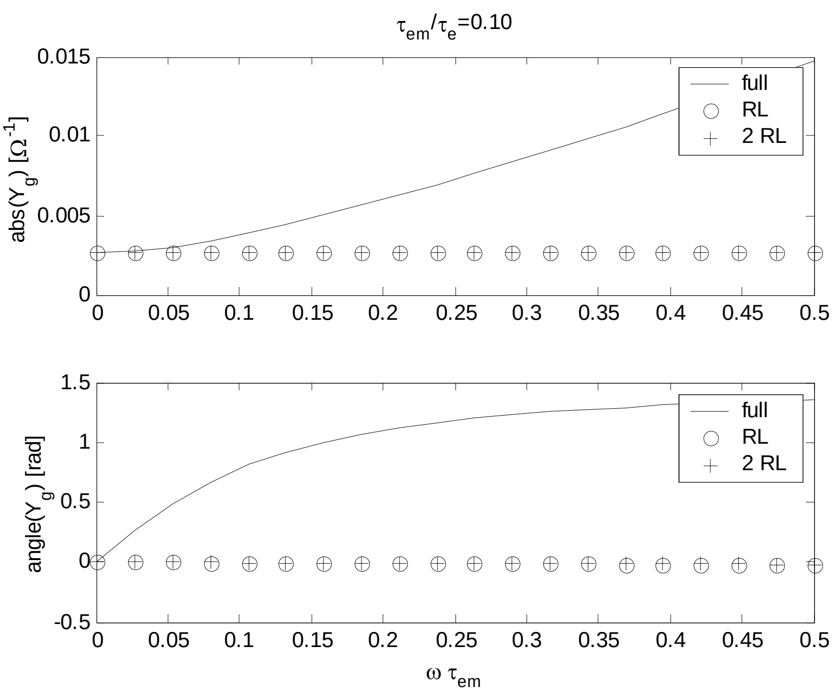
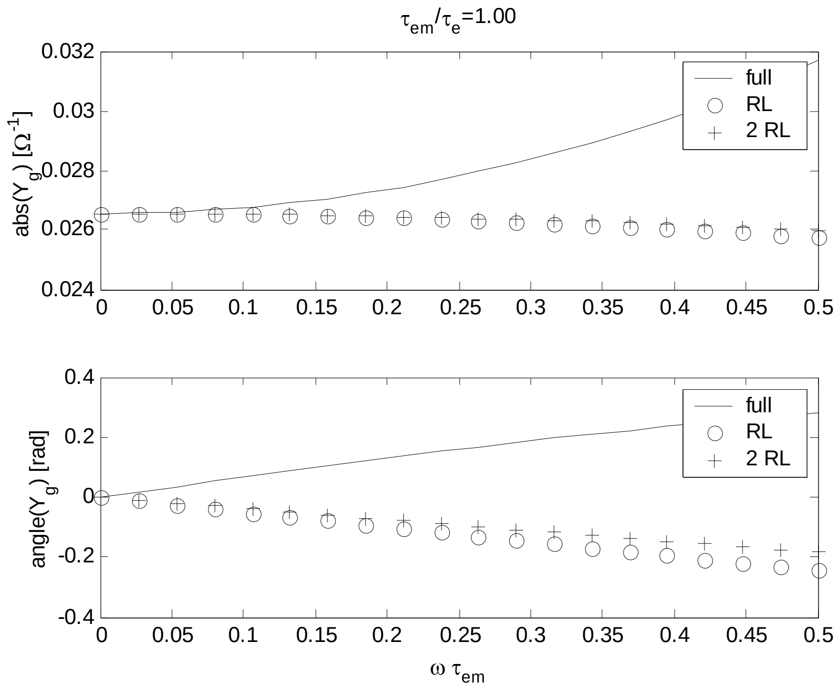


Fig. 5

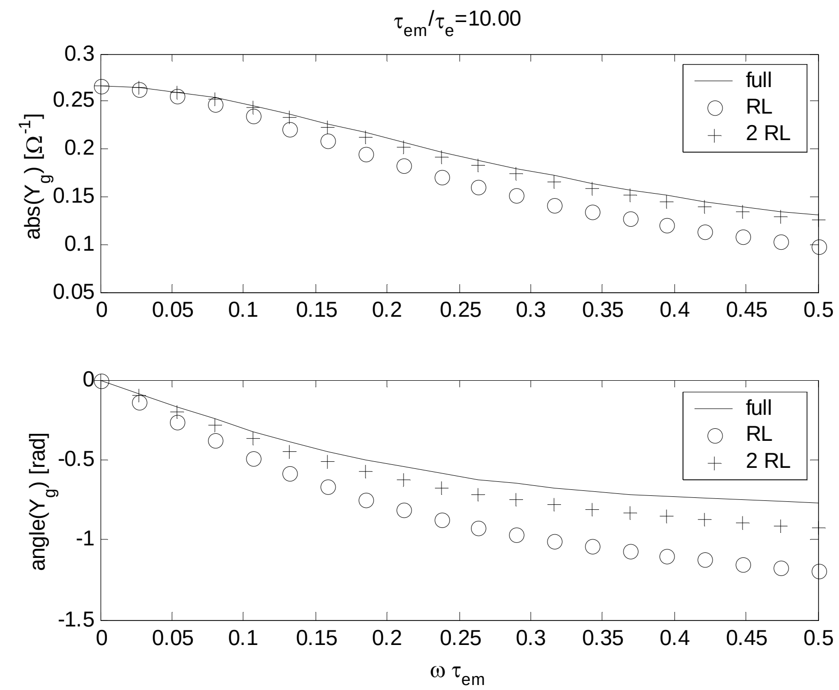
In Fig. 6 we report the behaviours of the frequency response of the circuit of Fig. 4 and 5 in terms of current, as compared with the true response given by (53b) with z=-l. The parameters used are the same as before. We can observe that when it results $\tau_{em}/\tau_e >> 1$ then the equivalent circuit of fig. 5 provides the correct answer, consistently with the condition assumed to derive this model.
We further observe that neither this circuit nor the RC circuit of Fig. 2 can reproduce correctly the true response of the circuit when $\tau_{em}/\tau_e = 1$, even for small values of $\omega\tau_{em}$.



(a)



(b)



(c)

Fig. 6: frequency responces

CASE C: $\dfrac{1}{\tau_m} \approx \dfrac{1}{\tau_{em}} \approx \dfrac{1}{\tau_e}$

In this case, the equations to be solved are the full equations (43), whose solution has already been determined. However, we are always in the case in which $\omega\tau_{em}$.<<1 $\Rightarrow$ $|\beta\, l|$<<1, so that we can repeat on the solution (53) all the approximations applied in the previous case, and get the tee equivalent circuit of Fig.7.

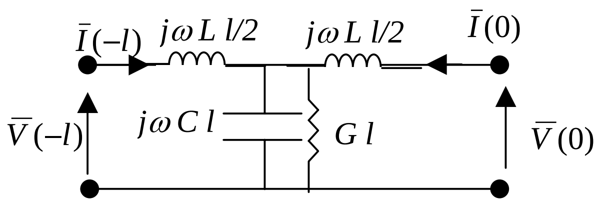
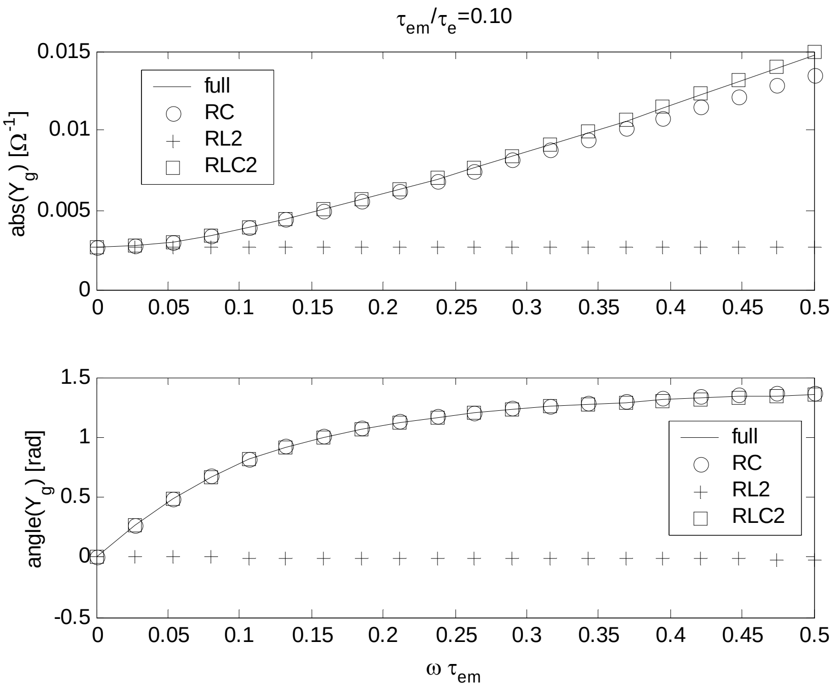


Fig. 7

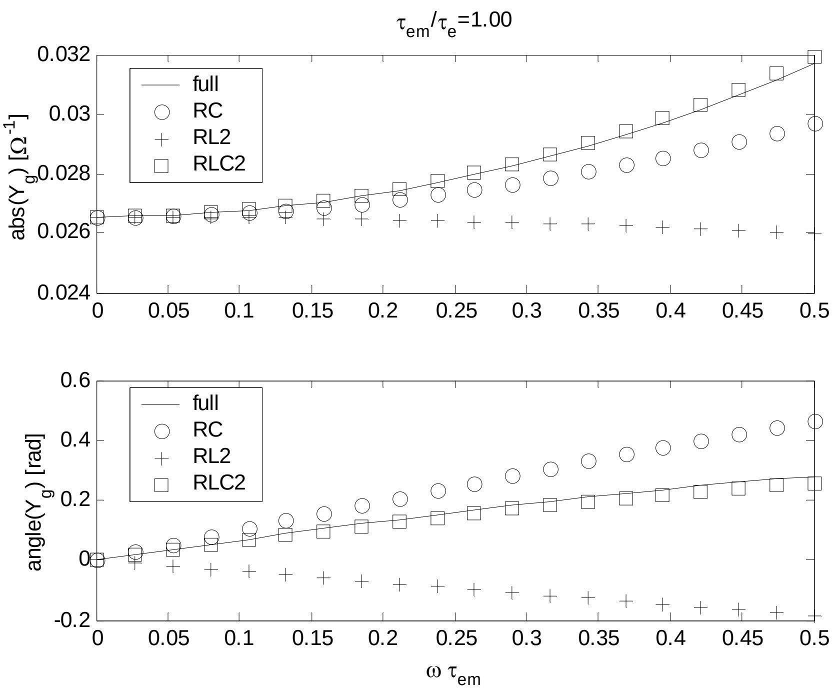
Of course, also in this case we could split the system in two and get the cascade of two circuits like the one depicted in Fig. 7.

In Fig. 8 we report the behaviours of the frequency response of the circuit obtained with two tee cells, as compared with the true response given by (53b) with z=-l, and the results obtained with the previous equivalent circuits. The geometrical parameters used are the same as before. We can observe that this circuit is able to reproduce correctly the results also when it results $\tau_{em}/\tau_e = 1$, as expected.



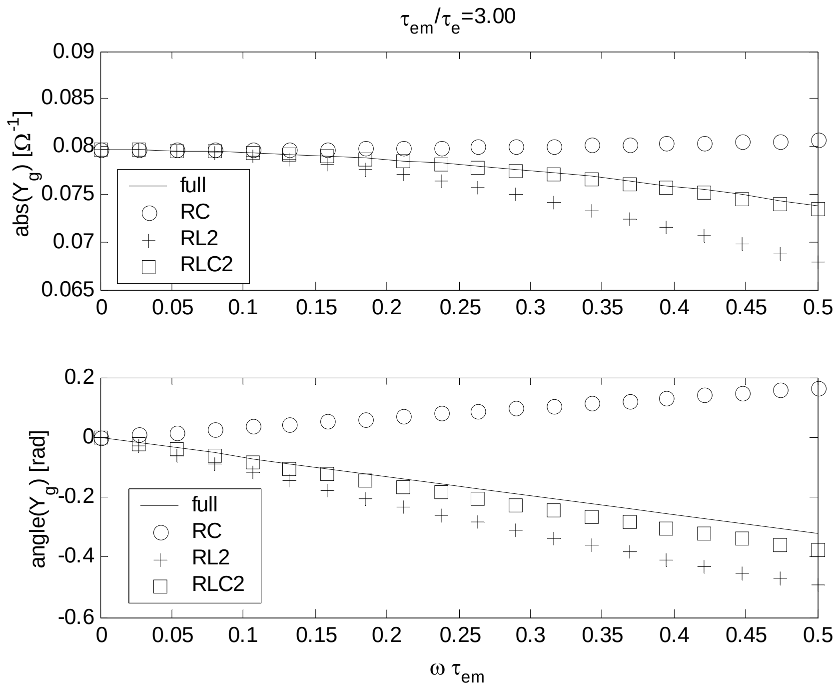
τ_{em}/τ_e=0.10
abs(Y_g) [Ω^{-1}]
angle(Y_g) [rad]
full
RC
RL2
RLC2
$\omega\ \tau_{em}$

(a)

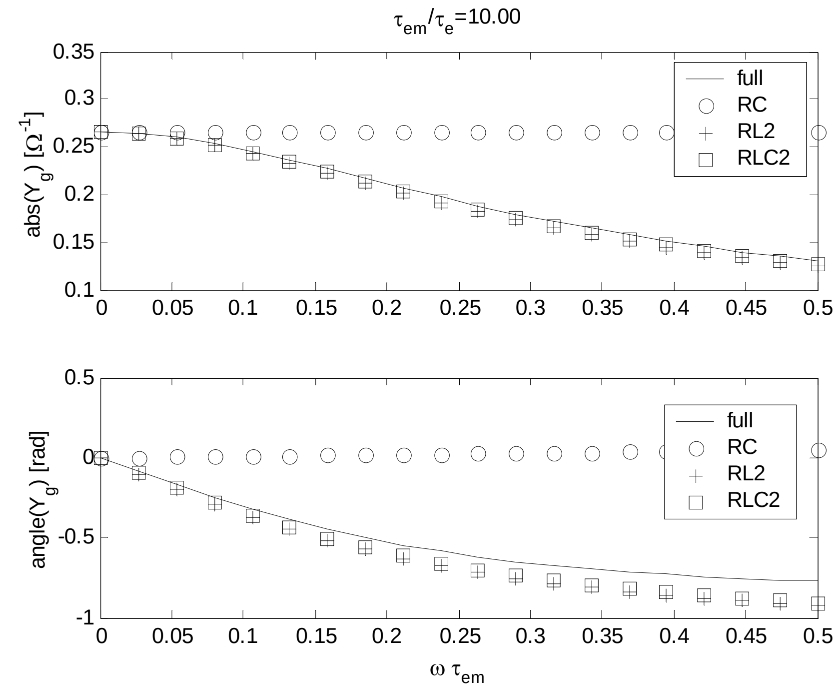


τ_{em}/τ_e=1.00
abs(Y_g) [Ω^{-1}]
angle(Y_g) [rad]
full
RC
RL2
RLC2
$\omega\ \tau_{em}$

(b)



(c)



(d)

Fig. 8: frequency responses

[1] H. A. Haus, J. R. Melcher, Electromagnetic fields and energy, Prentice Hall, Englewood Cliffs, New Jersey, 1989.
[2] L. De Menna, G. Miano, Unpublished