Notes on Circuital Representation of Two-Conductor Transmission Lines

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3.1 Ideal Lines

3.2 Lossy RLGC Lines

Introduction

Let us consider the two-conductor transmission line sketched in Figure 3.1a. The line imposes a well-defined relation between the terminal voltages and currents, since the solution of transmission line equations is unique when, besides the initial conditions, the voltage or the current at each line end is imposed. Instead, no solution would be found if one tried to impose more than two variables, for example, as well as the voltages at both ends, also the current at one end. Obviously these relations depend only on the transmission line equations and not on the elements to which the line is connected.

As a consequence, the interaction between the transmission line and the rest of the circuit in which it is inserted can be analyzed by describing the line through an equivalent two-port, as shown in Figure 3.1b.

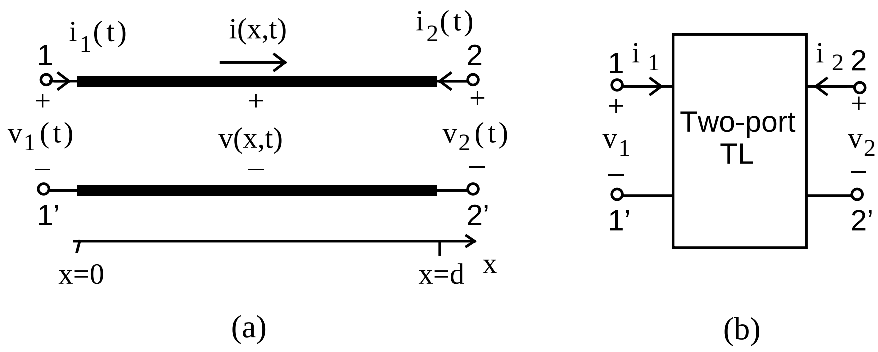

Figure 3.1 Two-conductor transmission line as a two-port element

In this Chapter we will derive time-domain equivalent two-ports describing two-conductor transmission lines. This will be done for any kind of lines: ideal lines, lossy RLGC lines, lossy lines with parameters depending on frequency, lines with space-varying parameters (nonuniform lines).

The representation of a line as a two-port in time domain is a basic step that can present difficulties. The degree of difficulty obviously depends on the nature of the line. The solution may be found directly in time domain only when dealing with ideal lines. The most general case of imperfect lines will be analyzed by means of Laplace (or Fourier) transforms and convolution theorem, by exploiting the assumption to deal with linear and time-invariant lines.

As we know from the circuit theory, there are six possible explicit representations of two of the four terminal variables in terms of the remaining two (e.g., Chua, Desoer and Kuh, 1987): the *current-controlled voltage-controlled, hybrid* and *transmission* representations. However, as well known, these *input-output descriptions* are not suitable when a time-domain analysis has to be carried out (*e.g.*, Miano and Maffucci, 2000).

Other representations are possible, based on the consideration that transmission lines are systems with an *internal state*, hence *input-state-output* descriptions may be introduced for them. Several *input-state-output* descriptions are possible, depending on the choice of the state variables. Due to the linearity, the propagation along the transmission line can always be represented through the superposition of two *travelling waves*: a *forward wave* propagating toward the right and a *backward wave* propagating toward the left. As we shall show, the input-state-output description obtained by choosing the travelling waves as state variables, has a comprehensible physical meaning and leads to a mathematical model that is at the same time elegant, extremely simple and effectively solvable by means of numerical recursive algorithms.

The two-port representing a two-conductor transmission line can be always modeled by a Thévenin equivalent circuit of the type shown in Figure 3.2, whatever the nature of the transmission line. The behavior of each end of the transmission line is described through a linear time-invariant one-port connected in series with a linear controlled voltage source. The choice of the traveling waves as state variables yields two fundamental properties that make particularly useful this representation:

a) the impedance at each port is the driving-impedance when the other port is matched;

b) the control laws describing the controlled sources are of delayed type.

Property b), that is a direct consequence of the propagation phenomenon, is of considerable importance, since it allows the controlled voltage sources to be treated as *independent sources* if the problem is solved by means of an iterative procedure, as done when using simulators like SPICE.

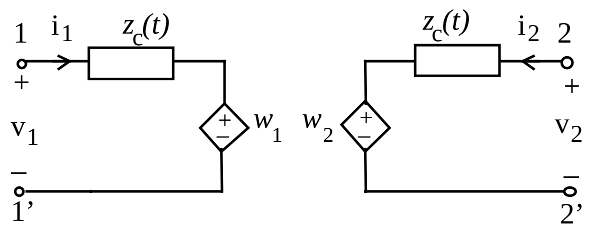

Figure 3.2 Equivalent circuit of Thévenin type of a two-conductor transmission line

To determine the impulse responses characterizing the transmission line as two-port, we have to solve the line equations with the proper initial conditions by considering the currents at the line ends as if they were known. The degree of difficulty in solving a problem of this kind, obviously, will depend on the nature of the line.

3.1 Ideal lines

3.1.1 The d'Alembert Solution of the Line Equations

The equations for ideal two-conductor transmission lines are, (see Chapter 2),

$$-\frac{\partial v}{\partial x}=L\frac{\partial i}{\partial t} \;, \qquad -\frac{\partial i}{\partial x}=C\frac{\partial v}{\partial t} \;, \tag{3.1.1}$$

where L and C are positive constants.

The general solution of Eqs. (3.1.1) may be found by operating directly in the time domain. Starting from these equations, by derivation and substitution, we obtain the following system of uncoupled equations

$$\frac{\partial^2 v}{\partial x^2}-LC\frac{\partial^2 v}{\partial t^2}=0 \;, \qquad \frac{\partial^2 i}{\partial x^2}-LC\frac{\partial^2 i}{\partial t^2}=0 \;. \tag{3.1.2}$$

Each of these has the form of a *wave equation (*e.g., Smirnov, 1964a). By placing

$$c=\frac{1}{\sqrt{LC}} \;, \tag{3.1.3}$$

the general solution of (3.1.2) in d'Alembert form is

$$v(x,t)=v^{+}(t-x/c+\alpha^{+})+v^{-}(t+x/c+\alpha^{-}), \tag{3.1.4}$$

$$i(x,t)=i^{+}(t-x/c+\alpha^{+})+i^{-}(t+x/c+\alpha^{-}), \tag{3.1.5}$$

where α^{+} and α^{-} are arbitrary constants. The functions v^{+}, v^{-}, i^{+} and i^{-} are arbitrary. Generally they can be continuous or indeed generalized functions in the sense of the theory of distributions, for example, unit step functions, Dirac functions, etc., (e.g., Courant and Hilbert, 1989).

The term $v^{+}(t-x/c+\alpha^{+})$ represents a *traveling voltage wave*, which propagates in the positive direction of the x axis, with constant velocity *c*, without being distorted. It is the so-called *forward voltage wave*. Similarly, $v^{-}(t+x/c+\alpha^{-})$ is a *traveling voltage wave* that propagates in the direction of negative x. It is the so-called *backward voltage wave*. Analogous considerations hold for i^{+} and i^{-}, which are called, respectively, *forward current wave* and *backward voltage wave*.

The general solution may be equivalently represented through the superposition of two *standing waves* (e.g., Collin, 1992), but this decomposition is useless for our purposes (*e.g.*, Miano and Maffucci, 2000).

The set of all the possible solutions of Eqs. (3.1.2) is much ampler than the set of the solutions of the original system (3.1.1). As (3.1.4) and (3.1.5) are solutions of the wave equations, it is sufficient to ensure that they satisfy any of the two equations of the original system. Substituting (3.1.4) and (3.1.5) in (3.1.1) we obtain

$$i^{+} = \frac{v^{+}}{R_c}, \; i^{-} = -\frac{v^{-}}{R_c}, \tag{3.1.6}$$

where

$$R_c = \sqrt{L/C} \tag{3.1.7}$$

is the *characteristic "resistance"* of the line. Finally, substituting (3.1.6) and (3.1.7) in (3.1.5) we obtain

$$i(x,t) = \frac{1}{R_c}\left[v^{+}(t - x/c + \alpha^{+}) - v^{-}(t + x/c + \alpha^{-})\right]. \tag{3.1.8}$$

In this way, the general solution of the line equations is represented in terms of forward and backward voltage waves only. Clearly we can also represent it through the superposition of forward and backward current waves. We need only to use (3.1.6) and (3.1.7).

Let us consider now a line of finite length *d*, and let us assume

$$\alpha^{+} = 0 \;, \quad \alpha^{-} = -d \;. \tag{3.1.9}$$

As a consequence, at time instant *t*, $v^{+}(t)$ represents the amplitude of the forward voltage wave at left end of the line, $x = 0$, and while $v^{-}(t)$ represents the amplitude of the backward voltage wave at right end, $x = d$. The time necessary for a wave leaving one end of the line to reach the other end is the *one-way transit time*

$$T = d/c \tag{3.1.10}$$

The amplitudes of these waves are determined by imposing the initial distribution of the voltage and current along the line, that is, the *initial conditions*

$$i(x,t=0) = i_0(x) \quad v(x,t=0) = v_0(x) \quad \text{for} \;\; 0 \le x \le d \;, \tag{3.1.11}$$

and the *boundary conditions*, that are imposed by the circuital elements whom the line is connected to.

It can be easily shown that there is only one solution of the two-conductor transmission line equations compatible with assigned initial conditions and boundary values at each line end for the voltage or current (Miano and Maffucci, 2000). However, in general, the values of the voltage and current at line ends are not known, but are themselves unknowns of the problem: they depend on the actual network in which the line lies. The forward wave and the backward one interact mutually only through the circuits connected to the line.

3.1.2 Input-State-Output Description and the Equivalent Circuits of Thévenin and Norton Type

The voltages and currents at line ends are related to the voltage and current distributions along the line through the relations (see Figure 3.1 for the reference directions)

$$v_1(t) = v(x=0,t), \quad i_1(t) = i(x=0,t) \;, \tag{3.1.12}$$

$$v_2(t) = v(x=d,t), \quad i_2(t) = -i(x=d,t) \;. \tag{3.1.13}$$

To determine the relations between the variables v_1, v_2, i_1, and i_2 imposed by the transmission line we first impose the initial conditions along the line, and then particularize the solution of the line equations at the line ends. The general solution of the line equations is

$$v(x,t) = v^+(t - x/c) + v^-(t + x/c - T) \;, \tag{3.1.14}$$

$$i(x,t) = \frac{1}{R_c}\left[v^+(t - x/c) - v^-(t + x/c - T)\right], \tag{3.1.15}$$

where T is given by (3.1.10).

The voltage and current distributions along the line are completely identified by the functions $v^+(t)$ and $v^-(t)$, and vice versa, hence these functions completely specify the *state* of the line. We shall consider them as *state variables* of the line[1].

The initial conditions fix the state of the line v^+ and v^- in the time interval $(0,T)$. By placing

$$v_0^+(t) = \frac{1}{2}\{v_0[c(T-t)] + R_c i_0[c(T-t)]\} \qquad 0 \le t \le T \;, \tag{3.1.16}$$

$$v_0^-(t) = \frac{1}{2}\{v_0(ct) - R_c i_0(ct)\} \qquad 0 \le t \le T \;, \tag{3.1.17}$$

and imposing that expressions (3.1.14) and (3.1.15) satisfy the initial conditions (3.1.16) and (3.1.17), we obtain

$$v^+(t) = v_0^+(t) \text{ and } v^-(t) = v_0^-(t) \text{ for } \quad 0 \le t \le T \;. \tag{3.1.18}$$

The state of the line for $t > T$ depends on the values of the voltage and current at the line ends.

[1] It is easy to verify that the per-unit-length energy w_{em} associated to the electromagnetic field along the guiding structure is $w_{em}(x,t) = Li^2/2 + Cv^2/2 = C\left[v^+(t - x/c)\right]^2 + C\left[v^-(t + x/c - T)\right]^2$, while the electrical power absorbed by an ideal line may be expressed as $i_1 v_1 + i_2 v_2 = \left\{\left[v^+(t+T)\right]^2 + \left[v^-(t+T)\right]^2 - \left[v^+(t)\right]^2 - \left[v^-(t)\right]^2\right\}/R_c$

Specifying expressions (3.1.14) and (3.1.15) at end $x=0$, we obtain relations

$$v_1(t)=v^+(t)+v^-(t-T) \tag{3.1.19}$$

$$R_c i_1(t)=v^+(t)-v^-(t-T) \tag{3.1.20}$$

whereas specifying them at end $x=d$, we obtain relations

$$v_2(t)=v^+(t-T)+v^-(t), \tag{3.1.21}$$

$$-R_c i_2(t)=v^+(t-T)-v^-(t). \tag{3.1.22}$$

Subtracting (3.1.19) and (3.1.20) termwise, and summing (3.1.21) and (3.1.22), termwise, we have, respectively, for $t>T$:

$$v_1(t)-R_c i_1(t)=2v^-(t-T), \tag{3.1.23}$$

$$v_2(t)-R_c i_2(t)=2v^+(t-T). \tag{3.1.24}$$

If the state of the line were completely known at any *t*, these equations would completely determine the terminal behavior of the line.

Different formulations of the equations governing the state dynamics for $t>T$ are possible. From Eqs. (3.1.19) and (3.1.21) we immediately obtain

$$v^+(t)=v_1(t)-v^-(t-T) \text{ for } t>T, \tag{3.1.25}$$

$$v^-(t)=v_2(t)-v^+(t-T) \text{ for } t>T, \tag{3.1.26}$$

whereas from Eqs. (3.1.20) and (3.1.22) we immediately obtain

$$v^+(t)=R_c i_1(t)+v^-(t-T) \text{ for } t>T, \tag{3.1.27}$$

$$v^-(t)=R_c i_2(t)+v^+(t-T) \text{ for } t>T. \tag{3.1.28}$$

Instead, summing (3.1.19) and (3.1.20) and subtracting (3.1.21) and (3.1.22) we have:

$$v^+(t)=\frac{1}{2}[v_1(t)+R_c i_1(t)] \text{ for } t>0, \tag{3.1.29}$$

$$v^-(t)=\frac{1}{2}[v_2(t)+R_c i_2(t)] \text{ for } t>0. \tag{3.1.30}$$

Eqs. (3.1.25) and (3.1.26) (or Equation (3.1.27) and (3.1.28)) describe in *implicit form* the relation between the state of the line and the electrical variables at the line ends, whereas Eqs. (3.1.29) and (3.1.30) provide the same relation, but in an *explicit form*.

Remarks

a) From Eqs. (3.1.29) and (3.1.30) we immediately deduce a very important property of ideal two-conductor transmission lines: v^+ would be equal to zero if the line were connected at the left end to a resistor with resistance R_c - *matched line at the left end*. The same considerations hold for the backward wave if the line is *matched at the right end*.

b) Eqs. (3.1.23) and (3.1.24), joined to the equations describing the state dynamics, give, in implicit form, the relations between the voltages v_1, v_2 and the currents i_1, i_2. Relations (3.1.23) and (3.1.24) are two fundamental results of considerable importance. Equation (3.1.23) says that the voltage at any time t at end $x=0$ is equal to the sum of two terms: the term $R_c i_1(t)$, that would be the if the backward voltage wave were equal to zero, and the term $2v^-(t-T)$, that is the voltage that would appear if the line were connected to an open circuit at end $x=0$. The factor 2 agrees with the following consideration: when the current at left end is zero, the amplitude of the forward wave is equal to that of backward one, and then the voltage is two times the amplitude of v^-. If the backward voltage wave v^- were known for any $t>T$, the behavior of the line at the left end would be entirely determined by (3.1.23). The backward voltage wave is known for $0 \le t \le T$ from the initial conditions, whereas for $t>T$ it depends on both what is connected to the right and left ends of the line, due to the reflections, and so it is an unknown of the problem. Similar considerations can be given to (3.1.24).

c) What is the physical meaning of the equations describing the state dynamics? Let us consider, for example, the state variable v^+. Equation (3.1.25) (or (3.1.27)) describes the dynamics of v^+ in an implicit form, whereas Eq. (3.1.29) describes the dynamics of v^+ in an explicit form. Eqs. (3.1.25), (3.1.27) and (3.1.29) all state the same property: the amplitude of the forward voltage wave at time t and at the left end of the line is equal to the amplitude of the same wave at the right end and at the previous time instant $t-T$ (for any $t>T$). This is one of the fundamental properties of an ideal two-conductor transmission line. Similar considerations hold for the equations of the other state variable v^-. Thus, for $t>T$ the values of state variables at the time instant t only depend on the values of the voltages and/or currents at the line ends, and of the state variables themselves at the time instant $t-T$.

d) Two different descriptions of the characteristic relations of the two-port representing the line are possible. The set of Eqs. (3.1.23)-(3.1.26) describe the terminal properties of the line, as well as the internal state. We call them the *internal* or *input-state-output description* of the line. Eqs. (3.1.25) and (3.1.26) govern the dynamics of the state, whereas Eqs. (3.1.23) and (3.1.24) describe the terminal properties. Therefore we call Eqs. (3.1.25) and (3.1.26) *state equations*, and Eqs. (3.1.23) and (3.1.24) "*output equations*". A description in which only the terminal variables are involved is possible. In fact by substituting the expression of v^+ given by (3.1.29) into relation (3.1.24), and the expression of v^- given by (3.1.30) into relation (3.1.23), we eliminate the state variables. This formulation of the characteristic relations of the two-port representing the line is called the *external* or *input-output description* (*e.g.*, Miano and Maffucci, 2000) ♦

It is useful to rewrite Eqs. (3.1.23) and (3.1.24), and state Eqs. (3.1.25) and (3.1.26) in terms of the new state variables w_1 and w_2 defined, for $t > T$, as follows

$$w_1(t) \equiv 2v^-(t-T) \ , \tag{3.1.31}$$

$$w_2(t) \equiv 2v^+(t-T) \ . \tag{3.1.32}$$

Then equations representing the line behavior at the ends can be rewritten as

$$v_1(t) - R_c i_1(t) = w_1(t), \tag{3.1.33}$$

$$v_2(t) - R_c i_2(t) = w_2(t) \ . \tag{3.1.34}$$

For $0 \le t \le T$, the state variables w_1 and w_2 only depends on the initial conditions of the line through

$$w_1(t) = 2v_0^-(t) \quad \text{for} \quad 0 \le t \le T \ , \tag{3.1.35}$$

$$w_2(t) = 2v_0^+(t) \quad \text{for} \quad 0 \le t \le T \ . \tag{3.1.36}$$

For $t > T$, $w_1(t)$ and $w_2(t)$ are related, respectively, to the values of the voltages v_2 and v_1 at the time instant $t - T$, and to the values of the state variables w_2 and w_1 at the time instant $t - T$ through the equations

$$w_1(t) = [2v_2(t-T) - w_2(t-T)] \quad \text{for} \quad t > T \ , \tag{3.1.37}$$

$$w_2(t) = [2v_1(t-T) - w_1(t-T)] \quad \text{for} \quad t > T \ . \tag{3.1.38}$$

Eqs. (3.1.37) and (3.1.38) are linear algebraic difference relations with one delay, which have to be solved with the initial conditions (3.1.35) and (3.1.36).

Eqs. (3.1.33) and (3.1.34) suggest the equivalent circuit of Thévenin type shown in Figure 3.1.1a. Each port of the line behaves as a linear resistor of resistance R_c connected in series with a controlled voltage source. The state Eqs. (3.1.37) and (3.1.38) govern the controlled voltage sources w_1 and w_2. This equivalent circuit was proposed for the first time by Branin (Branin, 1967).

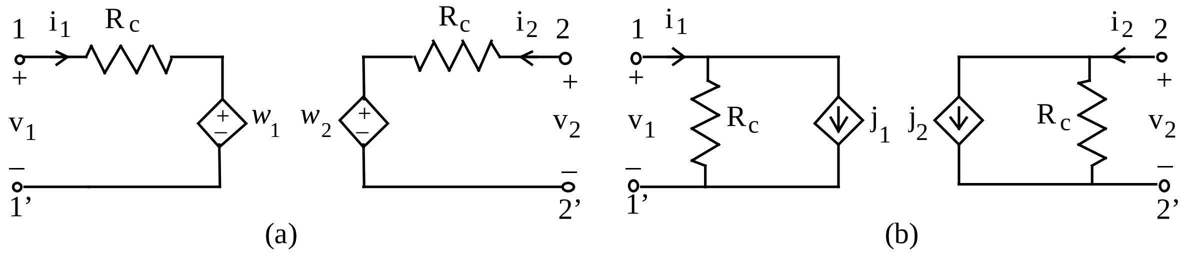

Figure 3.1.1. Time domain equivalent circuit of an ideal line: Thévenin (a), and Norton type (b)

Figure 3.1.1b shows an equivalent circuit of Norton type. The controlled current sources $j_1(t)$ and $j_2(t)$ are related to the controlled voltage sources through the relations

$$j_1 = -\frac{w_1}{R_c}\,,\quad j_2 = -\frac{w_2}{R_c}\,. \tag{3.1.39}$$

The governing laws of the these sources may be expressed, for $t > T$, as follows

$$j_1(t) = [-2i_2(t-T) + j_2(t-T)], \tag{3.1.40}$$

$$j_2(t) = [-2i_1(t-T) + j_1(t-T)]\,. \tag{3.1.41}$$

For $0 \le t \le T$ they are expressed in terms of the initial conditions

$$j_1(t) = -2v_0^-(t)/R_c \qquad j_2(t) = -2v_0^+(t)/R_c\,. \tag{3.1.42}$$

3.1.3 Lines with Distributed Sources

Until now we have referred to ideal two-conductor transmission lines without distributed sources. Here we show how the two-port model obtained in the previous paragraph can be extended to lines with distributed sources. The equations for these lines are

$$\frac{\partial v}{\partial x} + L\frac{\partial i}{\partial t} = e_s, \qquad \frac{\partial i}{\partial x} + C\frac{\partial v}{\partial t} = j_s\,. \tag{3.1.43}$$

The general solution of these equations may be expressed in the form

$$v(x,t) = v^{+}(t - x/c) + v^{-}(t + x/c - T) + v_p(x,t) \ , \tag{3.1.44}$$

$$i(x,t) = \frac{1}{R_c}\left[v^{+}(t - x/c) - v^{-}(t + x/c - T)\right] + i_p(x,t) \ , \tag{3.1.45}$$

where $v_p(x,t)$ and $i_p(x,t)$ is a particular solution for system (3.1.43). One possible solution is the one that satisfies null initial conditions and the boundary conditions of the line as if the line were perfectly matched. This ensures particular solutions of short duration. To determine this solution let us use Green's function method.

Applying the property of sampling of the Dirac function, we rewrite system (3.1.43) as

$$\frac{\partial v}{\partial x} + L\frac{\partial i}{\partial t} = \int_{0^-}^{t^+} \delta(t-\tau) e_s(x,\tau) d\tau \ , \tag{3.1.46}$$

$$\frac{\partial i}{\partial x} + C\frac{\partial v}{\partial t} = \int_{0^-}^{t^+} \delta(t-\tau) j_s(x,\tau) d\tau \ . \tag{3.1.47}$$

Let us indicate with $h_v^{(e)}(x,t;\tau)$ and $h_i^{(e)}(x,t;\tau)$ the solution of the "auxiliary problem" defined by

$$\frac{\partial h_v^{(e)}}{\partial x} + L\frac{\partial h_i^{(e)}}{\partial t} = \delta(t-\tau) e_s(x,\tau) \ , \tag{3.1.48}$$

$$\frac{\partial h_i^{(e)}}{\partial x} + C\frac{\partial h_v^{(e)}}{\partial t} = 0 \ , \tag{3.1.49}$$

the initial conditions

$$h_v^{(e)}(x,t=0;\tau) = 0 \quad h_i^{(e)}(x,t=0;\tau) = 0 \quad \text{for } 0 \le x \le d \ , \tag{3.1.50}$$

and the boundary conditions, for $t > 0$

$$h_v^{(e)}(x=0,t;\tau) = -R_c h_i^{(e)}(x=0,t;\tau) , \tag{3.1.51}$$

$$h_v^{(e)}(x=d,t;\tau) = R_c h_i^{(e)}(x=d,t;\tau) . \tag{3.1.52}$$

Moreover, let $h_v^{(j)}(x,t;\tau)$ and $h_i^{(j)}(x,t;\tau)$ be the solution of the equations

$$\frac{\partial h_v^{(j)}}{\partial x} + L\frac{\partial h_i^{(j)}}{\partial t} = 0 \ , \tag{3.1.53}$$

$$\frac{\partial h_i^{(j)}}{\partial x} + C\frac{\partial h_v^{(j)}}{\partial t} = \delta(t-\tau) j_s(x,\tau) \ , \tag{3.1.54}$$

with the same initial and boundary conditions for $h_v^{(e)}$ and $h_i^{(e)}$. Using the superposition property, a particular solution of Eqs. (3.1.43) can be expressed in the form

$$v_p(x,t) = \int_0^t \left[h_v^{(e)}(x,t;\tau) + h_v^{(j)}(x,t;\tau)\right] \mathrm{d}\tau \ , \tag{3.1.55}$$

$$i_p(x,t) = \int_0^t \left[h_i^{(e)}(x,t;\tau) + h_i^{(j)}(x,t;\tau)\right] \mathrm{d}\tau \ . \tag{3.1.56}$$

This particular solution is, by construction, equal to zero at the initial time and satisfies the perfect matching conditions at the line ends.

Let us begin to resolve the first auxiliary problem. We soon observe that for $0 \le t \le \tau^-$ the solution is identically null and, for $t \ge \tau^+$, Eqs. (3.1.47) and (3.1.48) become homogeneous; $\tau^- = \tau - \varepsilon$ and $\tau^+ = \tau + \varepsilon$ where ε is a positive and arbitrarily small number. So if we know the state of the line at $t = \tau^+$, we can solve the problem by using d'Alembert's solution, starting from that time instant.

Let us assume that function $e_s(x,t)$ is continuous and derivable with respect to *x*. We integrate both the sides of (3.1.47) and (3.1.48) in the time from $t = \tau - \varepsilon$ to $t = \tau + \varepsilon$. Using initial conditions and taking the limit $\varepsilon \to 0$ we determine the state of the line at time instant $t = \tau^+$,

$$h_i^{(e)}\left(x, t = \tau^+; \tau\right) = \frac{1}{L} e_s(x,\tau) \ , \tag{3.1.57}$$

$$h_v^{(e)}\left(x, t = \tau^+; \tau\right) = 0 \ . \tag{3.1.58}$$

Having assumed that $e_s(x,t)$ is continuous and derivable in respect to *x*, the terms $\partial h_v^{(e)} / \partial x$ and $\partial h_i^{(e)} / \partial x$ are limited and therefore do not contribute. Since the solutions we are looking for have to be equal to zero for $t < \tau$, we consider d'Alembert solutions of the form

$$h_v^{(e)}(x,t;\tau) = v^{+(e)}[x - c(t-\tau);\tau]u(t-\tau) + v^{-(e)}[x + c(t-\tau);\tau]u(t-\tau), \tag{3.1.59}$$

$$h_i^{(e)}(x,t;\tau) = \frac{1}{R_c} v^{+(e)}[x - c(t-\tau);\tau]u(t-\tau) - \frac{1}{R_c} v^{-(e)}[x + c(t-\tau);\tau]u(t-\tau). \tag{3.1.60}$$

The unknowns $v^{+(e)}(\xi;\tau)$ and $v^{-(e)}(\xi;\tau)$ are defined for $-\infty < \xi < +\infty$. Imposing conditions (3.1.57) and (3.1.58) and boundary conditions (3.1.50) and (3.1.51) we obtain

$$-v^{-(e)}(\xi;\tau) = v^{+(e)}(\xi;\tau) = \begin{cases} \dfrac{R_c}{2L} e_s(\xi,\tau) & \text{for } 0 \le \xi \le d \,, \\ 0 & \text{otherwise}. \end{cases} \tag{3.1.61}$$

Proceeding in the same way, we determine the solution of the other auxiliary problem. Assuming that the function $j_s(x,t)$ is continuous and derivable in respect to *x*, one has

$$h_i^{(j)}\left(x,t=\tau^+;\tau\right)=0\ , \tag{3.1.62}$$

$$h_v^{(j)}\left(x,t=\tau^+;\tau\right)=\frac{1}{C}\,j_s(x,\tau)\ . \tag{3.1.63}$$

Thus the solution to the second auxiliary problem can be expressed as

$$h_v^{(j)}(x,t;\tau)=v^{+(j)}[x-c(t-\tau);\tau]u(t-\tau)+v^{-(j)}[x+c(t-\tau);\tau]u(t-\tau), \tag{3.1.64}$$

$$h_i^{(j)}(x,t;\tau)=\frac{1}{R_c}v^{+(j)}[x-c(t-\tau);\tau]u(t-\tau)-\frac{1}{R_c}v^{-(j)}[x+c(t-\tau);\tau]u(t-\tau), \tag{3.1.65}$$

where the functions $v^{+(j)}$ and $v^{-(j)}$ are given by

$$v^{+(j)}(\xi;\tau)=v^{-(j)}(\xi;\tau)=\begin{cases}\dfrac{1}{2C}\,j_s(\xi,\tau) & \text{for } 0\le\xi\le d\,,\\ 0 & \text{otherwise}\,.\end{cases} \tag{3.1.66}$$

Specifying (3.1.44) and (3.1.45) at the end $x=0$, and subtracting them termwise, we obtain

$$v_1(t)-R_c i_1(t)=w_1(t)+e_1(t)\ , \tag{3.1.67}$$

where $w_1(t)$ has been defined in (3.1.31) and

$$e_1(t)=v_p(x=0,t)-R_c i_p(x=0,t)\ . \tag{3.1.68}$$

Specifying the expressions (3.1.44) and (3.1.45) at end $x=d$, and summing them termwise we have

$$v_2(t)-R_c i_2(t)=w_2(t)+e_2(t)\ , \tag{3.1.69}$$

where $w_2(t)$ has been defined in (3.1.32) and

$$e_2(t)=v_p(x=d,t)+R_c i_p(x=d,t)\ . \tag{3.1.70}$$

Thus it is clear that a line with distributed sources can be represented with the equivalent two-port of Thévenin type shown in Figure 3.1.1a, provided that two independent voltage sources are inserted that supply voltages $e_1(t)$ and $e_2(t)$, in series with the controlled voltage sources $w_1(t)$ and $w_2(t)$, respectively. The equations governing the dynamics of the state are

$$w_1(t)=2v_2(t-T)-w_2(t-T)-v_p(x=d,t-T)\ , \tag{3.1.71}$$

$$w_2(t)=2v_1(t-T)-w_1(t-T)-v_p(x=0,t-T)\ . \tag{3.1.72}$$

3.2. Lossy *RLGC* lines

This Section deals with lossy two-conductor transmission lines whose parameters are uniform in space and independent of frequency (RLGC lines). For such lines the general solution can not be expressed in the d'Alembert form because of the dispersion due to the losses. Several methods exist to approach this problem (e.g., Smirnov, 1964a; Doetsch, 1974), however, since we are assuming linear and time-invariant transmission lines, it is convenient to use the Laplace transform and the convolution theorem. In Laplace domain it is easy to solve the line equations, because they become ordinary differential equations. Once equivalent two-port representations of the line are obtained in Laplace domain, the time domain two-ports are immediately derived by applying the convolution theorem. Another feature of this method is the possibility to deal with lossy multiconductor lines with parameters depending on the frequency and nonuniform transmission lines, as we shall see later.

3.2.1. Time-Domain Analysis: Dispersion and Heaviside Condition.

In time domain the equations for lossy transmission lines are (see Chapter 2),

$$-\frac{\partial v}{\partial x} = L\frac{\partial i}{\partial t} + Ri \ , \qquad -\frac{\partial i}{\partial x} = C\frac{\partial v}{\partial t} + Gv \ , \tag{3.2.1}$$

where *L*, *C*, *R*, and *G* are positive constant parameters.

The system of the Eqs. (3.2.1) may be transformed into a system of two uncoupled second order partial differential equations as we have done for ideal transmission line equations. Starting from the Eqs. (3.2.1), by derivation and substitution we obtain the following equation for the voltage distribution

$$\frac{\partial^2 v}{\partial t^2} - c^2\frac{\partial^2 v}{\partial x^2} + (\alpha+\beta)\frac{\partial v}{\partial t} + \alpha\beta v = 0 \ , \tag{3.2.2}$$

where

$$c^2 = \frac{1}{LC}, \alpha = \frac{R}{L}, \beta = \frac{G}{C} \ ; \tag{3.2.3}$$

c would be the propagation velocity of the quasi-TEM mode if the line were lossless, α the inductive and β the capacitive damping factors. The current distribution satisfies a similar equation. If we introduce a new unknown function $u(x,t)$ (e.g., Smirnov, 1964a) such that

$$v(x,t) = e^{-\mu t}u(x,t) \ , \tag{3.2.4}$$

and

$$\mu = \tfrac{1}{2}(\alpha + \beta) = \tfrac{1}{2}\left(\frac{R}{L} + \frac{G}{C}\right), \tag{3.2.5}$$

we obtain the simpler equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = v^2 u \quad , \tag{3.2.6}$$

where

$$\nu = \tfrac{1}{2}(\alpha - \beta) = \tfrac{1}{2}\left(\frac{R}{L} - \frac{G}{C}\right). \tag{3.2.7}$$

Note that it is always $0 \le |\nu| \le \mu$.

Let us look for a harmonic wave solution of Eq. (3.2.6), of the type

$$u(x,t) = U\cos(\omega t - \gamma x) \cdot \tag{3.2.8}$$

The dispersion relation is given by

$$\omega^2 + \nu^2 = c^2\gamma^2, \tag{3.2.9}$$

and the phase velocity is

$$c_{ph} = \frac{\omega}{\gamma} = c\frac{\omega}{\sqrt{\omega^2 + \nu^2}}. \tag{3.2.10}$$

Since the phase velocity c_{ph} depends upon the frequency, any signal is propagated with distortion. Equation (3.2.6) reduces to the dispersionless wave equation (see Section 3.1)) if, and only if,

$$\nu = 0 \text{, that is, } \alpha = \beta \ . \tag{3.2.11}$$

This is the so-called *Heaviside condition* that Oliver Heaviside discovered in 1887, while studying the possibility of distortionless transmission of telegrapher signals. In this case the solution of Eq. (3.2.3) is an "undistorted" travelling wave:

$$v^{\pm}(x,t) = e^{-\mu t} u(t \mp x/c) \tag{3.2.12}$$

Where u is an arbitrary function. The losses only cause a damping of the wave. This result has been important for telegraphy. It shows that, given appropriate values for the per-unit-length parameters of the line, signals can be transmitted in an undistorted form, even if damped in time. "*The distortionlessless state forms a simple and natural boundary between two diverse kind of propagation of a complicated nature, in each of which there is continuous distortion ...* " (Heaviside, 1893).

When $\nu \neq 0$, the general solution of Eq. (3.2.6) can not be expressed in the d'Alembert form because the phase velocity c_{ph} depends upon the frequency. For lines with frequency-dependent losses, the attenuation will be different for various frequency components and there will be a further variation in the phase velocity.

3.2.2 Solution of the line equations in Laplace domain

Let us consider transmission lines initially at rest. Non-zero initial conditions can be dealt with by using equivalent distributed sources along the line, as done for ideal lines in paragraph 3.13 (see Miano and Maffucci, 2000).

In Laplace domain system (3.2.1) reads

$$\frac{dV(x;s)}{dx} = -Z(s)I(x;s), \qquad \frac{dI(x;s)}{dx} = -Y(s)V(x;s), \tag{3.2.13}$$

where $V(x;s)$ is the Laplace transform of the voltage distribution, $I(x;s)$ is the Laplace transform of the current distribution and

$$Z(s) = R + sL \qquad Y(s) = G + sC \tag{3.2.14}$$

The parameters Z and Y are, respectively, the *per-unit-length longitudinal impedance and transverse admittance* of the line in the Laplace domain.

Starting from Eqs. (3.2.13), by derivation and substitution, we obtain the two uncoupled second order differential equations

$$\frac{d^2V}{dx^2} - k^2(s)V = 0 \qquad \frac{d^2I}{dx^2} - k^2(s)I = 0 \tag{3.2.15}$$

where we have introduced the function [1]

$$k(s) = \sqrt{Z(s)Y(s)} \; . \tag{3.2.16}$$

By substituting the expressions (3.2.14) in (3.2.16) we obtain

$$k(s) = \sqrt{(R+sL)(G+sC)} = \frac{s}{c}\sqrt{(1+\mu/s)^2 - (\nu/s)^2} \; , \tag{3.2.17}$$

---

[1] The expression $-ik(s = i\omega)$ is the so called *propagation constant.* For ideal transmission lines it is equal to ω/c.

This function is multivalued, having two branches (e.g., Smirnov, 1964b)[2].

As done for ideal lines in Section 3.1, the general solution of line Eqs. (3.2.15) may be written as a superposition of two *traveling waves*

$$V(x,s) = V^{+}(s)e^{-k(s)(x-x^{+})} + V^{-}(s)e^{k(s)(x-x^{-})}, \tag{3.2.18}$$

$$I(x,s) = I^{+}(s)e^{-k(s)(x-x^{+})} + I^{-}(s)e^{k(s)(x-x^{-})}, \tag{3.2.19}$$

where V^{+}, V^{-}, I^{+} and I^{-} are arbitrary functions and x^{+} and x^{-} are arbitrary constants. This form highligths forward and backward waves. By choosing $x^{+}=0$ and $x^{-}=d$, where *d* is the line length, it is easy to realize that V^{+} is the amplitude of the forward voltage wave at left end of the line, $x=0$, and while V^{-} represents the amplitude of the backward voltage wave at right end, $x=d$.

The solution may be equivalently expressed as a superposition of any other couple of independent solutions. For instance, we may choose the so-called *standing waves*, namely solutions having the form $\cos[k(s)x]$ and $sin[k(s)x]$ (Collin, 1992; Franceschetti, 1997).

Obviously, the set of all possible solutions for the Eqs. (3.2.15) is much ampler than the set of the solutions of the original system (3.2.13). As (3.2.18) and (3.2.19) are solutions of the Eqs. (3.2.15) it is necessary to ensure that they satisfy one of the two equations of the original system. For example, substituting (3.2.18) in (3.2.13) we obtain

$$I(x;s) = \frac{1}{Z_c(s)}\left[V^{+}(s)e^{-k(s)(x-x^{+})} - V^{-}(s)e^{k(s)(x-x^{-})}\right], \tag{3.2.20}$$

where

$$Z_c(s) = \sqrt{\frac{Z(s)}{Y(s)}} \tag{3.2.21}$$

is the *characteristic impedance* of the line.

[2] The function $k(s)$ has two first-order branch points of regular type along the negative real axis: $s_R = -R/L$ and $s_G = -G/C$. When the Heaviside condition $\nu = 0$ is satisfied, the two branch points coincide and hence cancel each other out. We can choose anyone of the two branches of $k(s)$ to determine the general solution of our problem. Hereafter we shall consider the branch that has positive real part for $\mathrm{Re}\{s\} > 0$, thus in the lossless limit $k \to +s/c$. In consequence we must operate in the domain C_{cut} obtained by cutting the complex plane along the segment belonging to the negative real axis whose ends are s_R and s_G . This branch of $k(s)$ has positive imaginary part for $\mathrm{Im}\{s\} > 0$.

By substituting (3.2.14) in the expression (3.2.21) we obtain for $Z_c(s)$[3]

$$Z_c(s)=\sqrt{\frac{R+sL}{G+sC}}=R_c\sqrt{\frac{1+(\mu+\nu)/s}{1+(\mu-\nu)/s}}\ , \tag{3.2.22}$$

where R_c is the *characteristic resistance*

$$R_c=\sqrt{\frac{L}{C}}\ . \tag{3.2.23}$$

For ideal transmission lines, $\mu=\nu=0$, and the expressions (3.2.17) and (3.2.22) reduce to $k=s/c$ and $Z_c=R_c$, respectively. The parameter R_c would be the characteristic resistance of the transmission line if it were without losses.

The arbitrary functions $V^+(s)$ and $V^-(s)$ have to be determined by imposing the *boundary conditions*. In Chapter 1 it is shown that there is only one solution of the transmission line Eqs. (3.2.13) compatible with an assigned voltage or current at each line end. Clearly we can also represent the solution through $I^+(s)$ and $I^-(s)$. It is easy to show that $I^+=V^+/Z_c$ and $I^-=-V^-/Z_c$.

Remark

e) The general solution (3.2.18) and (3.2.20) holds also in the inductiveless limit $L\to 0$ (or in the capacitiveless limit $C\to 0$). In this case we have $s_R\to 0$ (or $s_G\to 0$). The inductiveless limit $L\to 0$ is very important from the historical point of view. First successful submarine cable to transmit telegraph signals between England and France (1851) suggested the possibility of a transatlantic cable between Europe and United States. However, since the signal amplitude had been observed to fall off sharply with increasing length of the cable, Lord Kelvin (1855) studied the electrical transients in long cables assuming that the magnetic effects, described through the per-unit-length self-inductance *L*, were negligible. By using the circuit theory and the Kirchhoff's laws he derived a diffusion equation for the voltage for which Fourier (1822) had given solutions. In 1857 Kirchhoff extended the long line theory to include the effects of the self-inductance and deduced the finite velocity of propagation of the electrical signals. "However, Kelvin's theory of the cable dominated the thinking of everyone, perhaps because the extended theory looked unapproachable to physical interpretation" (Ernst Weber). In 1881 Heaviside reexamined the effect of the self-induction and determined what is now known as the "travelling wave" solution.♦

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[3] Function $Z_c(s)$ has two first-order branch points: $s_R=-R/L$ (*regular type*) and (*polar type*). However, according to the choice made for $k(s)$, in the domain C_{cut} function $Z_c(s)$ is single valued and its real part is always positive.

3.2.3 Terminal Behavior of the Line in Laplace Domain

By operating as done in Section 3.1 for ideal lines, it is possible to find an equivalent two-port representation of the line in Laplace domain, see Figure 3.2.1.

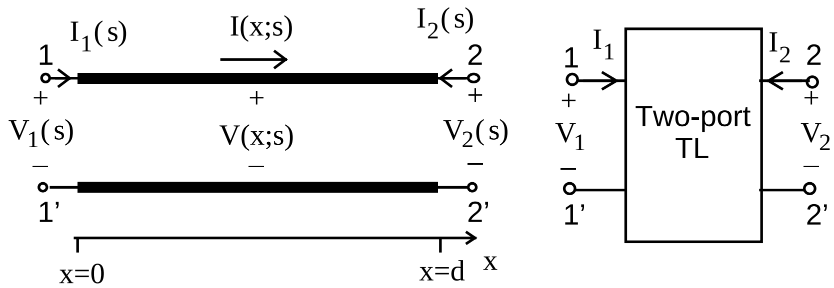

Figure 3.2.1 Two-conductor transmission line as a two-port element in Laplace domain.

The voltages and the currents at the line ends are related to the voltage and current distributions along the line through the relations

$$V_1(s)=V(x=0;s),\quad I_1(s)=I(x=0;s)\ , \tag{3.2.24}$$
$$V_2(s)=V(x=d;s),\quad I_2(s)=-I(x=d;s)\ . \tag{3.2.25}$$

Specifying the expressions (3.2.18) and (3.2.20) at the end $x=0$ we obtain relations

$$V_1(s)=V^+(s)+P(s)V^-(s)\ , \tag{3.2.26}$$
$$Z_c(s)I_1(s)=V^+(s)-P(s)V^-(s)\ , \tag{3.2.27}$$

where the *global propagation operator* $P(s)$ is

$$P(s)=\exp\left[-sT\sqrt{(1+\mu/s)^2-(\nu/s)^2}\right]\ , \tag{3.2.28}$$

and T is the one-way transit time of the quasi-TEM mode

$$T=\frac{d}{c}\ . \tag{3.2.29}$$

Instead, specifying the expressions (3.2.18) and (3.2.20) at the end $x=d$ we obtain relations

$$V_2(s)=P(s)V^+(s)+V^-(s)\ , \tag{3.2.30}$$
$$-Z_c(s)I_2(s)=P(s)V^+(s)-V^-(s)\ . \tag{3.2.31}$$

Subtracting (3.2.26) and (3.2.27), and summing (3.2.30) and (3.2.31), we have, respectively,

$$V_1(s)-Z_c(s)I_1(s)=2P(s)V^-(s)\ , \tag{3.2.32}$$
$$V_2(s)-Z_c(s)I_2(s)=2P(s)V^+(s)\ . \tag{3.2.33}$$

If the state of the line in the Laplace domain, represented by V^+ and V^-, were completely known, these equations would completely determine the terminal behavior of the line. Actually, as for the ideal two-conductor transmission lines, the voltage waves V^+ and V^- are themselves unknowns (see Section 3.1).

As for ideal transmission lines, different formulations of the equations governing the state are possible. By using (3.2.26) it is possible to express the amplitude of the outcoming forward wave at $x=0$, V^+, as a function of the voltage and amplitude of incoming forward wave at the same end. In the same way, by using (3.2.30) it is possible to express the amplitude of the outcoming backward wave at $x=d$, V^-, as a function of the voltage and amplitude of incoming forward wave at the same end. Therefore, from Eqs. (3.2.26) and (3.2.30) we immediately obtain

$$V^+(s)=V_1(s)-P(s)V^-(s), \tag{3.2.34}$$
$$V^-(s)=V_2(s)-P(s)V^+(s). \tag{3.2.35}$$

Instead, summing (3.2.26) and (3.2.27), termwise, and subtracting (3.2.30) and (3.2.31), termwise, we have

$$2V^+(s)=V_1(s)+Z_c(s)I_1(s), \tag{3.2.36}$$
$$2V^-(s)=V_2(s)+Z_c(s)I_2(s). \tag{3.2.37}$$

The state Equations (3.2.34) and (3.2.35) describe in *implicit form* the relation between the state of the line and the electrical variables at the line ends, whereas the state Equations (3.2.36) and (3.2.37) provide the same relation, but in an *explicit form*.

The Eqs. (3.2.32) and (3.2.33), joined to the state equations, give, in implicit form, the relations between the voltages V_1, V_2 and the currents I_1, I_2. In particular substituting the expression of V^+ given by (3.2.36) in the relation (3.2.33), and the expression of V^- given by (3.2.37) in the relation (3.2.32), we obtain two linearly independent equations in terms of the variables V_1, V_2, I_1 and I_2,

$$V_1(s)-Z_c(s)I_1(s)-P(s)[V_2(s)+Z_c(s)I_2(s)]=0, \tag{3.2.38}$$

$$V_2(s)-Z_c(s)I_2(s)-P(s)[V_1(s)+Z_c(s)I_1(s)]=0. \tag{3.2.39}$$

Remarks

a) The *global propagation operator* $P(s)$ defined by (3.2.28) plays an important role: it links the amplitude of the forward voltage wave at the line end $x=d$, PV^+, with the one at the line end $x=0$, V^+. Since the line is uniform, P is also the operator linking the amplitude of the backward voltage wave at the line end $x=0$, PV^-, with the one at the line end $x=d$, V^-. For nonuniform transmission lines, in general, they are different (see Section 3.4).

b) We immediately observe from Eq. (3.2.36) that V^+ would be equal to zero if the line were connected at the left end to a one-port with impedance $Z_c(s)$,

$$V_1(s)=-Z_c(s)I_1(s), \tag{3.2.40}$$

- *perfectly matched line at the left end* - and hence its inverse transform $v^+(t)$ should be zero. The same result holds for the backward wave if the line is *perfectly matched at the right end*,

$$V_2(s)=-Z_c(s)I_2(s), \tag{3.2.41}$$

In general, the matching conditions (3.2.40) and (3.2.41) are not satisfied. Then a forward wave with amplitude V^+ is generated at the left line end, $x=0$, and propagates toward the other end, $x=d$, where its amplitude is PV^+. Likewise, the backward wave excited at $x=d$ with amplitude V^- propagates toward the left line end, where its amplitude is PV^-. ♦

By operating in the Laplace domain, we have found, for the lossy two-conductor lines the same results as those we found for the ideal two-conductor lines, see Section 3.1. The system of Eqs. (3.2.32)-(3.2.35) and the system of Eqs. (3.2.38) and (3.2.39) are two fundamental results of considerable importance. They are two different mathematical models describing the two-port representing the line. The set of Eqs. (3.2.32)-(3.2.35) describe the internal state of the line, represented by V^+ and V^-, as well as the terminal properties. It is the *internal* or *imput-state-output description* of the line in the Laplace domain. Eqs. (3.2.34) and (3.2.35) govern the behavior of the state, whereas Eqs. (3.2.32) and (3.2.33) describe the terminal properties. Instead the system of Eqs. (3.2.38) and (3.2.39) describes only the terminal property of the line. It is the *external* or *input-output description* of the line in the Laplace domain.

3.2.4 Input-state-output description in Laplace domain: Thévenin and Norton equivalent circuits

As was done for ideal lines in Section 3.1, since in Eqs. (3.2.32) and (3.2.33) the state of the line appears through $2P(s)V^-(s)$ and $2P(s)V^+(s)$, we rewrite these equations as

$$V_1(s)-Z_c(s)I_1(s)=W_1(s), \tag{3.2.42}$$
$$V_2(s)-Z_c(s)I_2(s)=W_2(s), \tag{3.2.43}$$

where the state is represented by

$$W_1(s) = 2P(s)V^-(s)\,, \tag{3.2.44}$$
$$W_2(s) = 2P(s)V^+(s)\,. \tag{3.2.45}$$

Since the functions V^+, V^- are uniquely related to W_1, W_2, the latter functions can be regarded as state variables, too. Note that, except for the factor 2, $W_1(s)$ and $W_2(s)$ are, respectively, the backward voltage wave amplitude at the end $x=0$ and the forward voltage wave amplitude at the end $x=d$. From the state Equations (3.2.34) and (3.2.35) we obtain the equations for W_1 and W_2:

$$W_1(s) = P(s)[2V_2(s) - W_2(s)]\,, \tag{3.2.46}$$
$$W_2(s) = P(s)[2V_1(s) - W_1(s)]\,. \tag{3.2.47}$$

The *output equations* (3.2.42) and (3.2.43) say that, in Laplace domain, the voltage at each end of the line is equal to the sum of two terms: the one is due to the impedance Z_c, and the other to a controlled voltage source W. If the line were perfectly matched at end $x=d$, it would behave at $x=0$ as a one-port with impedance Z_c equal to $Z_c I_1$. Instead, W_1 is the voltage that there would be at the end $x=0$ if the line were connected to an open circuit at that end. Similar considerations can be made for the other end. In consequence, the behavior of each port of a lossy line may be represented through an equivalent Thévenin circuit, see Figure 3.2.2a. The governing laws of the controlled sources are nothing but the state equations (3.2.46) and (3.2.47). The Thévenin equivalent circuits of Figure 3.2.2a and the governing laws (3.2.46) and (3.2.47) for the voltage controlled sources still hold for lines with frequency-dependent parameters (see Section 3.3).

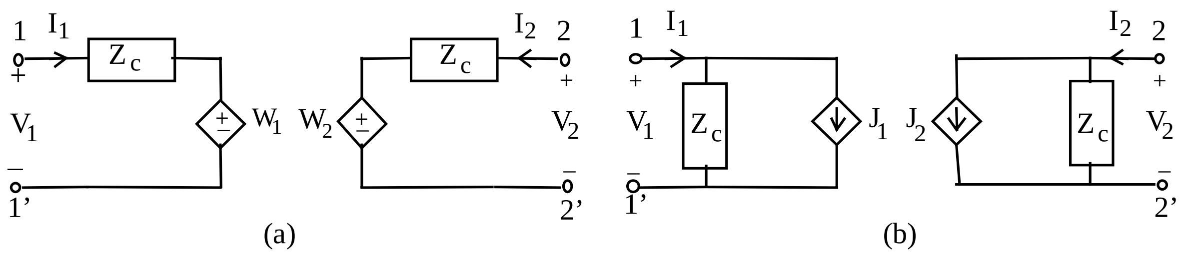

Figure 3.2.2 Laplace domain equivalent circuits for a uniform lossy two-conductor line: Thevénin (a) and Norton (b) type

Figure 3.2.2b shows the line equivalent circuit of Norton type in the Laplace domain. The controlled current sources $J_1(s)$ and $J_2(s)$ are related to the controlled voltage sources of the circuit 3.2.2a through

$$J_1 = -\frac{W_1}{Z_c} \quad \text{and} \quad J_2 = -\frac{W_2}{Z_c} \; . \tag{3.2.48}$$

The control laws of these sources may be obtained from the state equations (3.2.46) and (3.2.47). We obtain

$$J_1(s) = P(s)\left[-2I_2(s) + J_2(s)\right] \, , \tag{3.2.49}$$
$$J_2(s) = P(s)\left[-2I_1(s) + J_1(s)\right] \, . \tag{3.2.50}$$

Both the descriptions are completely characterized by the two functions $Z_c(s)$ and $P(s)$, which we call the *describing functions* of the two-port representing the line behavior in the Laplace domain. Next paragraph will be devoted to the study of the properties of such functions.

3.2.5 Properties of the describing functions $Z_c(s)$ and $P(s)$.

As shown in previous paragraph, the characteristic impedance $Z_c(s)$ is the impedance seen at one end of the line when the other end is matched. This is the case of a semi-infinite line (see Figure 3.2.3), that is *naturally* matched.

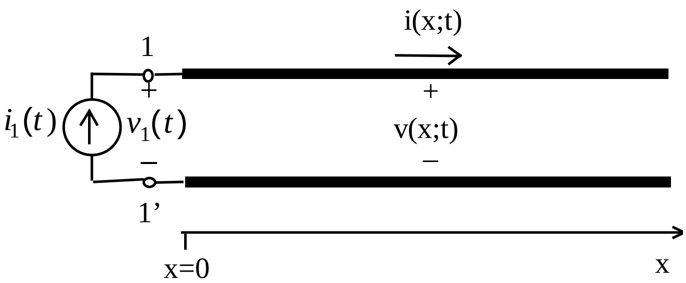

Figure 3.2.3 Semi-infinite line fed by an independent current source.

For such a line, placing

$$V_1(s) = V(x=0, s) \text{ and } I_1(s) = I(x=0, s) \, , \tag{3.2.51}$$

it is evident that

$$V_1(s) = Z_c(s) I_1(s) \; . \tag{3.2.52}$$

By applying the convolution theorem, relation (3.2.52), in the time domain, becomes

$$v_1(t) = (z_c * i_1)(t) \; . \tag{3.2.53}$$

where $z_c(t)$ is the inverse Laplace transform of $Z_c(s)$. The function $z_c(t)$ is the *current-controlled impulse response* of the one-port representing the line behavior at the end $x = 0$.

Since in the limit of lossless transmission line $Z_c(s) \to R_c$, it is useful to separate this term from that which takes into account the effects of the losses:

$$Z_c(s) = Z_{cp}(s) + Z_{cr}(s) \ , \tag{3.2.54}$$

where the *principal part* $Z_{cp}(s)$ and the *remainder* $Z_{cr}(s)$ are given by

$$Z_{cp}(s) = R_c \tag{3.2.55}$$

$$Z_{cr}(s) = R_c\left(\sqrt{\frac{1+(\mu+\nu)/s}{1+(\mu-\nu)/s}} - 1\right) \tag{3.2.56}$$

The impulse response $z_c(t)$ is given by

$$z_c(t) = R_c\delta(t) + z_{cr}(t) \ , \tag{3.2.57}$$

where $z_{cr}(t)$ is the inverse Laplace transform of $Z_{cr}(s)$ and is given by (Doetsch, 1970)

$$z_{cr}(t) = \nu R_c e^{-\mu t}\left[I_0(\nu t) + I_1(\nu t)\right]u(t), \tag{3.2.58}$$

where $I_0(\nu t)$ and $I_1(\nu t)$ are the modified Bessel functions of order 0 and 1, respectively, (e.g., Abramowitz and Stegun, 1972). Since $I_0(y)$ is an even function and $I_1(y)$ an odd function for real *y*, the term $z_{cr}(t)$ depends on the sign of ν. Two terms form the impulse response $z_c(t)$: a Dirac pulse acting at $t = 0$ and the *bounded* piecewise continuous function $z_{cr}(t)$ given by (3.2.58), which is equal to zero for $t < 0$. For ideal lines or lines in Heaviside condition $\nu = 0$, hence $z_{cr}(t) = 0$.

The remainder $Z_{cr}(s)$ has the following asymptotic behavior

$$Z_{cr}(s) \approx R_c \frac{\nu}{s} + O\left(1/s^2\right) \text{ for } s \to \infty , \tag{3.2.59}$$

whereas the principal part Z_{cp} is a constant. The inverse Laplace transform of Z_{cr} is a bounded piecewise continuous function because Z_{cr} goes to zero as $1/s$ for $s \to \infty$, whereas the inverse Laplace transform of the principal part is a Dirac function. Therefore, (3.2.54) is an *asymptotic expression* of $Z_c(s)$[4]

By substituting expression (3.2.57) in convolution relation (3.2.53) we obtain

$$v_1(t) = R_c i_1(t) + (z_{cr} * i_1)(t) \ . \tag{3.2.60}$$

---

[4] *"An asymptotic expression for a function is an expression as the sum of a simpler function and of a remainder that tends to zero at infinity, or (more generally) which tends to zero after multiplication by some power."* (Lighthill, 1958).

The first term is the voltage we should have at the end $x=0$ if the line were lossless or the Heaviside condition were satisfied. The other term describes the *wake* produced by the dispersion due to the losses.

The value of z_{cr} at $t=0^+$ is $z_{cr}(t=0^+)=\nu R_c$. The asymptotic behavior of z_{cr} for $t\to\infty$, is

$$z_{cr}(t)\approx\frac{2\nu R_c}{\sqrt{2\pi}}\frac{e^{-(G/C)t}}{\sqrt{|\nu|t}} \quad \text{for } \nu>0 \ , \tag{3.2.61}$$

$$z_{cr}(t)\approx\frac{\nu R_c}{2\sqrt{2\pi}}\frac{e^{-(R/L)t}}{(|\nu|t)^{3/2}} \quad \text{for } \nu<0 \ . \tag{3.2.62}$$

If $\mu\neq|\nu|$ the function z_{cr} goes exponentially to zero for $t\to\infty$ with the time constant $1/(\mu-|\nu|)$. Instead, in the limit case $\mu=|\nu|$, z_{cr} goes more slowly to zero for $t\to\infty$: as $1/\sqrt{\nu t}$ for $\nu>0$ and as $1/(|\nu|t)^{3/2}$ for $\nu<0$. Figure 3.2.4 shows the qualitative behavior of the function z_{cr} for different values of μ/ν.

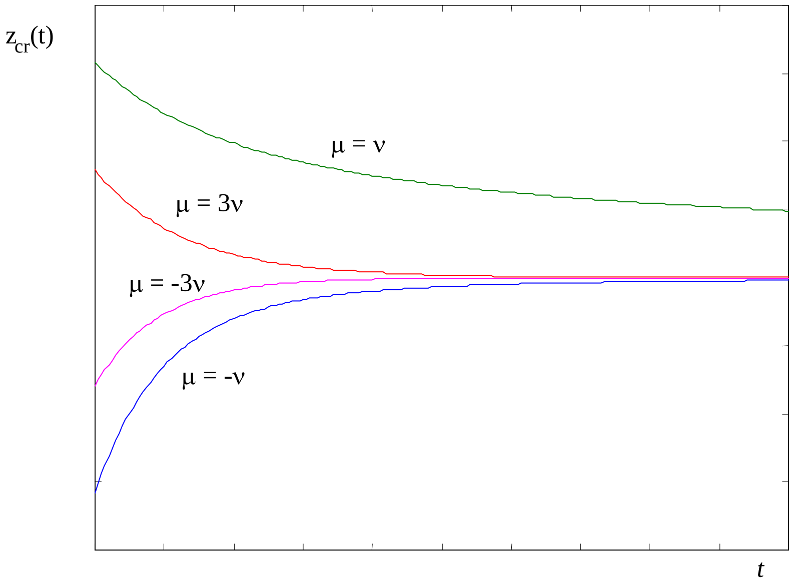

Figure 3.2.4. Qualitative behavior of the function $z_{cr}(t)$ for different values of μ/ν

Remark

The asymptotic behavior of $z_{cr}(t)$ for $t\to\infty$ depends only on the branch point of $Z_c(s)$ nearer to the imaginary axis.

For $\nu > 0$ we have $R/L > G/C$, so the branch point of polar type $-G/C$ is that nearer to the imaginary axis, and hence the factor $1/\sqrt{s+G/C}$ determines completely the asymptotic behavior of $z_{cr}(t)$. By using the inverse Laplace transform of $1/\sqrt{s}$ we immediately obtain that the inverse Laplace transform of $1/\sqrt{s+G/C}$ is the function $e^{-(G/C)t}/\sqrt{\pi t}$.

For $\nu = 0$ the two branch points coincide, and hence the function $Z_c(s)$ becomes a constant. Instead for $\nu < 0$ we have $R/L < G/C$, so the branch point of regular type $-R/L$ is that nearer to the imaginary axis, hence the factor $\sqrt{s+R/L}$ determines completely the asymptotic behavior of $z_{cr}(t)$. By using the inverse Laplace transform of $\sqrt{s}$ and the shifting property we realize that the inverse Laplace transform of $\sqrt{s+R/L}$ asymptotically behaves as $e^{-(R/L)t}/\left[\Gamma(-1/2)t^{3/2}\right]$ where $\Gamma(y)$ is the gamma function (e.g., Abramowitz and Stegun, 1972). ♦

Figure 3.2.5 shows the qualitative behavior of the voltage dynamics at the end $x = 0$, when $i_1(t)$ is a rectangular pulse of time length T and amplitude *I*.

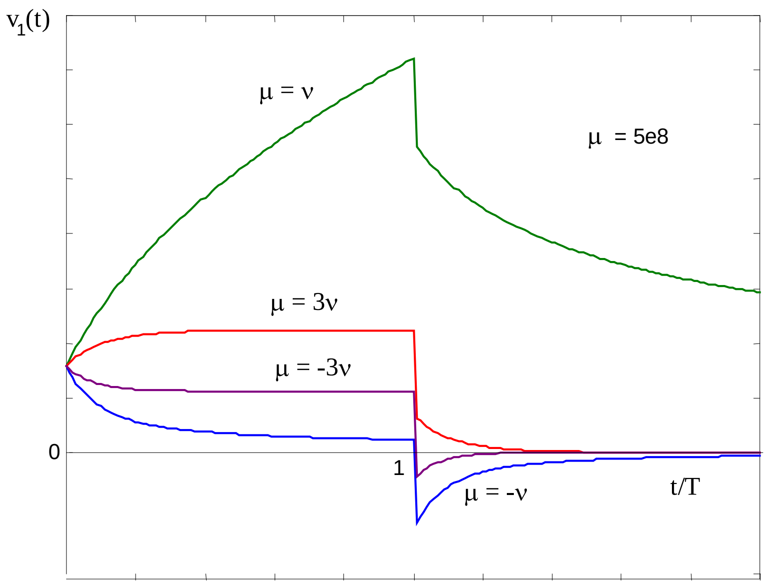

Figure 3.2.5 Voltage wave form at the end $x = 0$ of the semi-infinite line fed by is a rectangular pulse, for different values of μ/ν

As previously shown, the *global propagation operator* $P(s)$ defined by (3.2.28) relates the amplitude of the forward (backward) voltage wave at $x = d$ ($x = 0$) to the one at the line end $x = 0$ ($x = d$). It is useful to rewrite the propagation operator $P(s)$ as follows

$$P(s) = e^{-(s+\mu)T}\hat{P}(s)\,, \tag{3.2.63}$$

where the function $\hat{P}(s)$ is given by

$$\hat{P}(s) = \exp\left\{T(s+\mu)\left[1-\sqrt{1-\left(\frac{\nu}{\mu+s}\right)^2}\right]\right\}. \tag{3.2.64}$$

The factor $\exp[-(s+\mu)T]$, which represents a damped ideal delay operator with delay *T*, oscillates for $s \to \infty$, whereas the factor $\hat{P}(s)$ tends to 1. Therefore, the function $\hat{P}(s)$ has the following asymptotic expression

$$\hat{P}(s) = 1 + \hat{P}_r(s), \tag{3.2.65}$$

where the *remainder* $\hat{P}_r(s)$ has the following asymptotic behavior

$$\hat{P}_r(s) = \frac{\nu^2 T}{2}\frac{1}{s+\mu} + O\left(1/s^2\right) \text{ for } s \to \infty\,. \tag{3.2.66}$$

The first term of $\hat{P}(s)$, that is 1, is that which we should have if the line were lossless or if the Heaviside condition were satisfied. The other term, $\hat{P}_r(s)$, describes the wake produced by the temporal dispersion due to the losses. The operator *P* would coincide with the *ideal delay operator* $\exp(-sT)$ if the line were lossless. For lossy lines, the operator *P* reduces to the product of the ideal delay operator $\exp(-sT)$ with the decaying factor $\exp(-\mu T)$ when the Heaviside condition is satisfied. When $\nu \neq 0$ besides the ideal delay and the decaying factor, there is also the factor $\left(1+\hat{P}_r\right)$.

The inverse Laplace transform of *P* may be expressed as:

$$p(t) = e^{-\mu T}\delta(t-T) + p_r(t-T), \tag{3.2.67}$$

where the function

$$p_r(t) = T\nu e^{-\mu(t+T)}\frac{I_1\left[\nu\sqrt{(t+T)^2-T^2}\right]}{\sqrt{(t+T)^2-T^2}}u(t) \tag{3.2.68}$$

is the inverse Laplace transform of the function $P_r(s) = e^{-\mu T}\hat{P}_r(s)$. Therefore, the function $p(t)$ is equal to zero for $t < T$. This is the manifestation of the delay introduced by the finite value of the propagation velocity. According to the asymptotic behavior of $\hat{P}_r(s)$ given by (3.2.66), the function $p_r(t)$ is a bounded, piecewise continuous function. Since $I_1(y)$ is an odd function for real *y*, the function $p_r(t)$ does not depend on the sign of ν.

The first term on the right hand side of the expression (3.2.67) is the impulse response $p(t)$ that we should have if the Heaviside condition were satisfied. The remainder p_r describes the wake produced by the dispersion due to the losses. Figures 3.2.6 show the qualitative behavior of the function $p_r(t)$ for two values of *T*.

The value of $p_r(t)$ at time $t=0^+$ is $p_r(0^+)=T\nu^2/2$. For $t\to\infty$ the function $p_r(t)$ behaves as

$$p_r(t)\approx\frac{\nu^2 T}{\sqrt{2\pi}}\frac{e^{-(\mu-|\nu|)t}}{(|\nu|t)^{3/2}}. \tag{3.2.69}$$

Thus $p_r(t)$ vanishes exponentially for $t\to\infty$ with the time constant $1/(\mu-|\nu|)$ if $\mu\neq|\nu|$. Instead, it vanishes more slowly if $\mu=|\nu|$, as $1/(|\nu|t)^{3/2}$. Remember that for $\nu>0$ we have $\mu-|\nu|=G/C$ and for $\nu<0$ we have $\mu-|\nu|=R/L$.

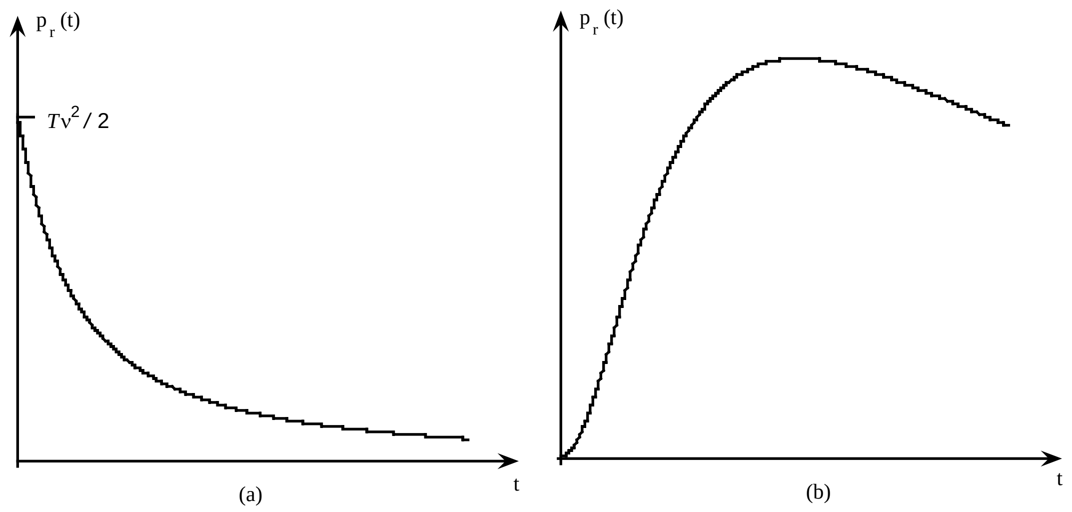

Figure 3.2.6 Qualitative behavior of the function $p_r(t)$,

for $T=T^*$ (a), and $T=10T^*$ (b), with $\mu=\nu$

Remark

As with $z_{cr}(t)$, the asymptotic behavior of $p_r(t)$ for $t\to\infty$ depends only on the branch point of $P(s)$ that is nearer to the imaginary axis. Unlike the branch points of $Z_c(s)$, both the branch points of $P(s)$ are regular.

For $\nu>0$ we have $R/L>G/C$, the branch point $-G/C$ is that nearer to the imaginary axis, and hence the factor $\sqrt{s+G/C}$ determines completely the asymptotic behavior of $p_r(t)$. By using the inverse Laplace transform of $\sqrt{s}$ we immediately obtain that the inverse Laplace transform behaves asymptotically as $e^{-(G/C)t}/\left[\Gamma(-1/2)t^{3/2}\right]$.

For $\nu = 0$ the two branch points coincide, and hence, they disappear. Instead for $\nu < 0$ we have $R/L < G/C$, to the branch point of regular type $-R/L$ is that nearer to the imaginary axis, and hence the factor $\sqrt{s + R/L}$ determines completely the asymptotic behavior of $p_r(t)$. By using the inverse Laplace transform of $\sqrt{s}$ and the linear transformation property we immediately obtain that the inverse Laplace transform behaves asymptotically as $e^{-(R/L)t}/\left[\Gamma(-1/2)t^{3/2}\right]$, where $\Gamma(y)$ is the gamma function (e.g., Abramowitz and Stegun, 1972). ♦

3.2.6 Input-State-Output Descriptions in Time Domain: Thévenin and Norton Equivalent Circuits.

The equivalent two-ports representing a lossy two-conductor line in time domain may are shown in Figure 3.2.7. The time domain output Equations of the two-port are obtained from the Eqs. (3.2.42) and (3.2.43), respectively

$$v_1(t) - (z_c * i_1)(t) = w_1(t), \tag{3.2.70}$$

$$v_2(t) - (z_c * i_2)(t) = w_2(t). \tag{3.2.71}$$

The voltages w_1 and w_2 only depend on the initial conditions for $0 \le t \le T$: $w_1(t) = w_2(t) = 0$ for $0 \le t \le T$, because we are considering transmission lines initially at rest. For $t > T$, they are both unknowns of the problem and are related to the voltage at the line ends through the control laws

$$w_1(t) = \{p * (2v_2 - w_2)\}(t), \tag{3.2.72}$$

$$w_2(t) = \{p * (2v_1 - w_1)\}(t). \tag{3.2.73}$$

Since $p(t) = 0$ for $0 \le t \le T$, Eqs. (3.2.72) and (3.2.73) reduce to

$$w_1(t) = u(t-T)\int_0^{t-T} p(t-\tau)\left[2v_2(\tau) - w_2(\tau)\right]d\tau, \tag{3.2.74}$$

$$w_2(t) = u(t-T)\int_0^{t-T} p(t-\tau)\left[2v_1(\tau) - w_1(\tau)\right]d\tau. \tag{3.2.75}$$

Therefore the voltages $w_1(t)$ and $w_2(t)$ depend, respectively, only on the values assumed by w_1 and w_2 and by v_1 and v_2 in the time interval $(0, t-T)$. Consequently, if the solution for $0 \le t \le iT$, with $i = 1, 2, \ldots$, is known, both w_1 and w_2 are known for $iT \le t \le (i+1)T$.

This fact allows the controlled sources w_1 and w_2 to be treated as "independent" sources if the problem is resolved by means of iterative procedures.

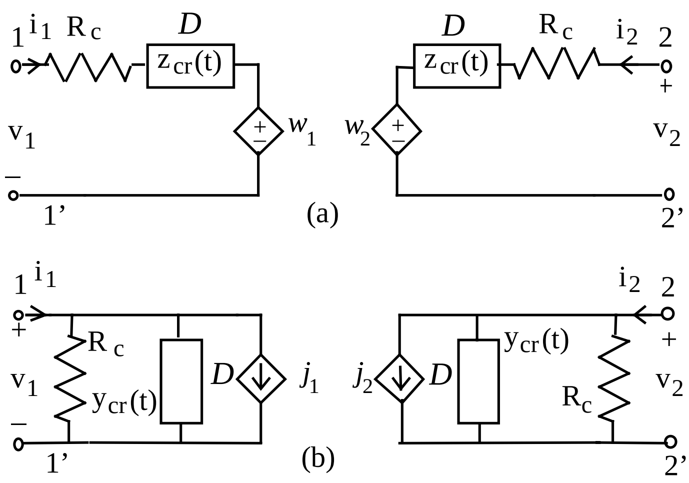

Figure 3.2.7 Time domain line equivalent circuit of Thévenin (a) and Norton type (b), for a uniform lossy two-conductor line

Unlike the lossless lines, due to the wake caused by the dispersion, $w_1(t)$ depends on the whole history of functions w_2 and v_2 in the interval $(0, t-T)$ and $w_2(t)$ depends on the whole history of functions w_1 and v_1 in the interval $(0, t-T)$.

By substituting (3.2.60) into Eqs. (3.2.70) and (3.2.71), we obtain the linear convolution Equations

$$v_1(t) - R_c i_1(t) - \{z_{cr} * i_1\}(t) = w_1(t) , \tag{3.2.76}$$

$$v_2(t) - R_c i_2(t) - \{z_{cr} * i_2\}(t) = w_2(t) . \tag{3.2.77}$$

These relations suggest the time domain equivalent circuit of Thévenin type shown in Figure 3.2.7a. The dynamic one-port is characterised by the current based impulse response $z_{cr}(t)$. It takes into account the wake due to the dispersion. When the Heaviside condition is satisfied we have $z_{cr}(t) = 0$, and the Thévenin equivalent circuit of Figure 3.2.7a reduces to that of Figure 3.1.1a, relevant to the lossless lines.

By substituting the expression of p given by (3.2.67) in the Eqs. (3.2.74) and (3.2.75), for the governing laws of the state we obtain the linear difference-convolution Equations

$$w_1(t) = [2v_2(t-T) - w_2(t-T)]e^{-\mu T} u(t-T) + \int_0^{t-T} p_r(t-T-\tau)[2v_2(\tau) - w_2(\tau)]d\tau, \tag{3.2.78}$$

$$w_2(t) = [2v_1(t-T) - w_1(t-T)]e^{-\mu T} u(t-T) + \int_0^{t-T} p_r(t-T-\tau)[2v_1(\tau) - w_1(\tau)]d\tau. \tag{3.2.79}$$

When the Heaviside condition is satisfied, we have $p_r(t)=0$, and the state Equations reduce to the linear difference Equations

$$w_1(t)=[2v_2(t-T)-w_2(t-T)]e^{-\mu T}u(t-T), \tag{3.2.80}$$

$$w_2(t)=[2v_1(t-T)-w_1(t-T)]e^{-\mu T}u(t-T). \tag{3.2.81}$$

These Equations, except for the damping factor $\exp(-\mu T)$, coincide with the state Equations obtained for the lossless lines, (see Section 3.1). Note that Eqs (3.2.70)-(3.2.73) still hold for lines with frequency-dependent parameters (see Section 3.3).

Figure 3.2.7b shows the time domain equivalent two-port of Norton type. In this circuit the dynamic one-port is characterised by the voltage based impulse response $y_{cr}(t)$. It is the bounded part of the inverse Laplace transform of the characteristic line admittance operator

$$Y_c(s)=\sqrt{\frac{G+sC}{R+sL}}=\frac{1}{R_c}\sqrt{\frac{1+(\mu-\nu)/s}{1+(\mu+\nu)/s}}. \tag{3.2.82}$$

The admittance Y_c can be rewritten as follows

$$Y_c(s)=\frac{1}{R_c}+Y_{cr}(s), \tag{3.2.83}$$

where the function Y_{cr} is given by

$$Y_{cr}(s)=\frac{1}{R_c}\left(\sqrt{\frac{1+(\mu-\nu)/s}{1+(\mu+\nu)/s}}-1\right) \tag{3.2.84}$$

and has the property

$$Y_{cr}(s)\approx-\frac{1}{R_c}\frac{\nu}{s}+O\left(1/s^2\right) \text{ for } s\to\infty . \tag{3.2.85}$$

The inverse Laplace transform of $Y_c(s)$ is given by

$$y_c(t) = \frac{1}{R_c}\delta(t) + y_{cr}(t), \tag{3.2.86}$$

where $y_{cr}(t)$ is the inverse Laplace transform of Y_{cr} and it is given by (see Doetsch, 1970)

$$y_{cr}(t) = \frac{\nu}{R_c} e^{-\mu t}\left[\mathrm{I}_1(\nu t) - \mathrm{I}_0(\nu t)\right]u(t). \tag{3.2.87}$$

Since I_0 is an even function and I_1 is an odd function (on the real axis), the term $y_{cr}(t)$ depends on the sign of the parameter ν. Because of the asymptotic behavior (3.2.85), the inverse Laplace transform of $Y_{cr}(s)$ is a bounded piece wise continuous function. Figures 3.2.8 show the qualitative behavior of the function $y_{cr}(t)$ for different values of μ/ν. The governing laws of the controlled current sources are

$$j_1(t) = \left[-2i_2(t-T) + j_2(t-T)\right]e^{-\mu T}u(t-T) + \{p_r * (-2i_2 + j_2)\}(t-T), \tag{3.2.88}$$

$$j_2(t) = \left[-2i_1(t-T) + j_1(t-T)\right]e^{-\mu T}u(t-T) + \{p_r * (-2i_1 + j_1)\}(t-T). \tag{3.2.89}$$

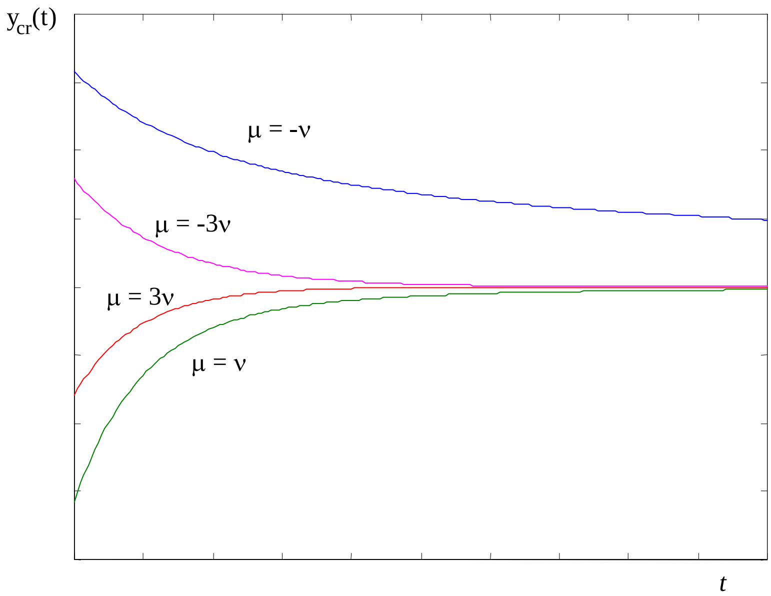

Figure 3.2.8 Qualitative behavior of the function $y_{cr}(t)$ for different values of μ/ν